

## PROBLEMS ON ALGEBRAIC TOPOLOGY

- The 2021-22 exam consisted of problems 1.2, 7.1, 4.1, 6.2 and 7.16.
- The 2022-23 exam consisted of problems 3.1, 2.1, 9.5 and 7.14.
- The 2023-24 exam consisted of problems 3.2, 2.2, 9.10 and 7.15.
- The 2024-25 exam consisted of problems 9.3 and 1.14.

### 1. HOMEOMORPHISMS

**Problem 1.1.** Let  $x$  be a point in  $\mathbb{R}^n$ , and suppose  $\epsilon > 0$ . Put  $U = \{y \in \mathbb{R}^n \mid \|x - y\| < \epsilon\}$ . In lectures we claimed that there is a homeomorphism  $f: U \rightarrow \mathbb{R}^n$  given by

$$f(y) = \frac{y - x}{1 - \|y - x\|^2/\epsilon^2} \quad f^{-1}(z) = x + \frac{\sqrt{\epsilon^2 + 4\|z\|^2} - \epsilon}{2\|z\|^2} \epsilon z.$$

Check carefully that these formulae give well-defined and continuous maps with the appropriate domains and ranges, and that they are inverse to each other.

**Solution:** Put  $U_1 = \{y \in \mathbb{R}^n \mid \|y\| < 1\}$ . We will first define a homeomorphism between  $U_1$  and  $\mathbb{R}^n$ . For  $y \in U_1$  we have  $1 - \|y\|^2 > 0$  so it is legitimate to divide by this quantity. We therefore have a continuous map  $f_1: U_1 \rightarrow \mathbb{R}^n$  given by  $f_1(y) = y/(1 - \|y\|^2)$ . Next, we can define  $g_1: \mathbb{R}^n \rightarrow \mathbb{R}^n$  by

$$g_1(z) = \begin{cases} 0 & \text{if } z = 0 \\ \frac{\sqrt{1+4\|z\|^2}-1}{2\|z\|^2} z & \text{if } z \neq 0. \end{cases}$$

In the second clause we have  $z \neq 0$  so  $2\|z\|^2 > 0$  so it is legitimate to divide by  $2\|z\|^2$ . It is clear that  $g$  is continuous on  $\mathbb{R}^n \setminus \{0\}$ , but continuity at 0 is more subtle.

Note that when  $z \neq 0$  we have

$$\begin{aligned} 1 - \|g_1(z)\|^2 &= 1 - \frac{2 + 4\|z\|^2 - 2\sqrt{1+4\|z\|^2}}{4\|z\|^2} \|z\|^2 \\ &= \frac{\sqrt{1+4\|z\|^2} - 1}{2\|z\|^2}. \end{aligned}$$

This is strictly positive, so  $\|g_1(z)\| < 1$ , so we can regard  $g_1$  as a map from  $\mathbb{R}^n$  to  $U_1$ . From the above formula it follows easily that  $f_1(g_1(z)) = z$  when  $z \neq 0$ , and it is clear that this also holds when  $z = 0$ .

Suppose instead that we start with  $y \in U_1 \setminus \{0\}$  and put  $z = f_1(y) = y/(1 - \|y\|^2) \neq 0$ . We then have

$$\begin{aligned} 1 + 4\|z\|^2 &= 1 + \frac{4\|y\|^2}{1 - 2\|y\|^2 + \|y\|^4} = \frac{1 + 2\|y\|^2 + \|y\|^4}{1 - 2\|y\|^2 + \|y\|^4} \\ &= \left( \frac{1 + \|y\|^2}{1 - \|y\|^2} \right)^2 \\ \sqrt{1 + 4\|z\|^2} - 1 &= \frac{1 + \|y\|^2}{1 - \|y\|^2} - 1 = \frac{2\|y\|^2}{1 - \|y\|^2} \end{aligned}$$

From this it follows that

$$g_1(f_1(y)) = \frac{2\|y\|^2(1 - \|y\|^2)^{-1}(1 - \|y\|^2)^{-1}y}{2\|y\|^2(1 - \|y\|^2)^{-2}} = y.$$

It is clear that this also holds when  $y = 0$ , so  $f_1$  and  $g_1$  are mutually inverse bijections.

Also, straightforward algebra shows that

$$\frac{\sqrt{1+4\|z\|^2} - 1}{2\|z\|} = \frac{2\|z\|}{\sqrt{1+4\|z\|^2} + 1} \leq \|z\|.$$

This proves that  $\|g_1(z)\| \leq \|z\|$ , so  $g_1$  is continuous at  $z = 0$ . It is clear that  $g_1$  is continuous at all other points, and  $f_1$  is continuous everywhere on  $U_1$ , so  $f_1$  and  $g_1$  are homeomorphisms.

Next, our original map  $f$  can be written as  $f_2 \circ f_1 \circ f_0$ , where  $f_0: U \rightarrow U_1$  is the homeomorphism given by  $f_0(y) = (y - x)/\epsilon$  and  $f_2: \mathbb{R}^n \rightarrow \mathbb{R}^n$  is the homeomorphism given by  $f_2(z) = \epsilon z$ . It follows that  $f$  is also a homeomorphism.

**Problem 1.2.** Consider a real vector space  $V$  with inner product, and a point  $x \in S(V)$ . Put  $A = \{y \in V \mid \langle y, x \rangle = 0\}$  and  $B = \{z \in S(V) \mid \langle z, x \rangle > 0\}$ . In the notes it is stated that there is a homeomorphism  $f: A \rightarrow B$  given by  $f(y) = (x + y)/\sqrt{1 + \|y\|^2}$ . Prove carefully that this is indeed a homeomorphism, and give a formula for the inverse. [Hint: you can start by simplifying  $\langle x, f(y) \rangle$ .]

**Solution:** First, for  $y \in A$  we have  $1 + \|y\|^2 \geq 1$  so there is no problem with taking the square root of this and dividing by it, so we have a well-defined continuous function  $f: A \rightarrow V$  given by  $f(y) = (x + y)/\sqrt{1 + \|y\|^2}$ . As  $x \in S(V)$  and  $\langle x, y \rangle = 0$  we have

$$\|x + y\|^2 = \langle x + y, x + y \rangle = \langle x, x \rangle + \langle y, y \rangle = \|x\|^2 + \|y\|^2 = 1 + \|y\|^2,$$

so  $\|f(y)\| = 1$ , so  $f(y) \in S(V)$ . We also have

$$\langle x, f(y) \rangle = \langle x, x + y \rangle / \sqrt{1 + \|y\|^2} = 1 / \sqrt{1 + \|y\|^2} > 0,$$

so  $f(y) \in B$ . We have therefore defined a continuous map  $f: A \rightarrow B$ .

Next, for  $z \in B$  we know that  $\langle x, z \rangle > 0$  so it is harmless to divide by  $\langle x, z \rangle$ . We can therefore define  $g: B \rightarrow V$  by  $g(z) = z / \langle x, z \rangle - x$ . This satisfies

$$\langle x, g(z) \rangle = \langle x, z \rangle / \langle x, z \rangle - \langle x, x \rangle = 1 - 1 = 0,$$

so we can regard  $g$  as a continuous map  $B \rightarrow A$ .

For  $y \in A$  we observed above that  $\langle x, f(y) \rangle = (1 + \|y\|^2)^{-1/2}$ ; from this it follows directly that  $f(y) / \langle x, f(y) \rangle = x + y$  and so  $g(f(y)) = y$ .

In the opposite direction, suppose we start with  $z \in B$  and put  $y = g(z) = z / \langle x, z \rangle - x$ . Using  $\|x\|^2 = \|z\|^2 = 1$  we get

$$1 + \|y\|^2 = 1 + \|z\|^2 / \langle x, z \rangle^2 + \|x\|^2 - 2\langle x, z \rangle / \langle x, z \rangle = \langle x, z \rangle^{-2}.$$

By the definition of  $B$  we have  $\langle x, z \rangle > 0$  so  $\sqrt{1 + \|y\|^2} = \langle x, z \rangle^{-1}$ . It follows easily that  $f(g(z)) = f(y) = z$ . This completes the proof that  $g$  is inverse to  $f$ , so  $f$  is a homeomorphism.

**Problem 1.3.** Recall that  $\mathfrak{u}(n) = \{\beta \in M_n(\mathbb{C}) \mid \beta + \beta^\dagger = 0\}$ . Find a basis for  $\mathfrak{u}(2)$  over  $\mathbb{R}$ , and prove that  $\mathfrak{u}(2)$  is not a complex vector subspace of  $M_2(\mathbb{C})$ .

**Solution:** The following matrices form a basis:

$$A_0 = \begin{bmatrix} i & 0 \\ 0 & 0 \end{bmatrix} \quad A_1 = \begin{bmatrix} 0 & 0 \\ 0 & i \end{bmatrix} \quad A_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad A_3 = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}.$$

As  $A_0 \in \mathfrak{u}(2)$  but  $iA_0 \notin \mathfrak{u}(2)$ , we see that  $\mathfrak{u}(2)$  is not a complex subspace.

**Problem 1.4.** Recall our definition of the lens space: we have a complex vector space  $V$  of dimension  $n$  with inner product, and an integer  $d > 1$ . We put  $C_d = \{\omega \in \mathbb{C} \mid \omega^d = 1\}$ , and we let this act on  $S(V)$  by multiplication. The lens space is then  $M = S(V)/C_d$ . What can you say in the special case where  $n = 1$ , or the special case where  $d = 2$ ?

**Solution:** When  $n = 1$  we can identify  $S(V)$  with  $S^1$ . A point of  $M$  is then a coset  $zC_d$  for some  $z \in S^1$ , and if  $zC_d = z'C_d$  then  $z' = z\omega$  for some  $\omega$  with  $\omega^d = 1$ , and thus  $z^d = (z')^d$ . There is thus a well-defined continuous map  $f: S^1/C_d \rightarrow S^1$  given by  $f(zC_d) = z^d$ , and this is in fact a homeomorphism.

In the case  $d = 2$ , we let  $P_{\mathbb{R}}(V)$  denote the space of one-dimensional real subspaces of  $V$ , and we define  $g: S(V)/C_2 \rightarrow P_{\mathbb{R}}(V)$  by  $g(vC_2) = \mathbb{R}v$ . This is easily seen to be a homeomorphism.

**Problem 1.5.** Let  $V$  be a finite-dimensional vector space with inner product. In Section 4 of the notes we defined spaces  $S(V_+)$ ,  $S'(V_+)$ ,  $S_+(V_+)/S(V)$ ,  $S^V$ , and  $B(V)/S(V)$ , and gave a table of formulae giving homeomorphisms between all these spaces. Verify a few of these formulae.

**Solution:** Put  $V_+ = \mathbb{R} \oplus V$  and

$$S^V = V \cup \{\infty\} = \text{the one-point compactification of } V$$

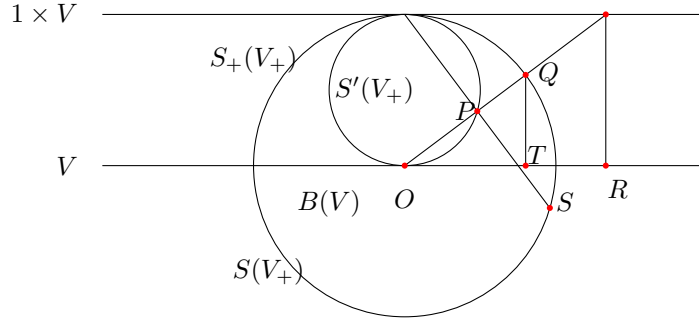
$$S(V_+) = \{(t, v) \in V_+ \mid t^2 + v^2 = 1\} \quad S_+(V_+) = \{(t, v) \in S(V_+) \mid t \geq 0\}$$

$$S'(V_+) = \{(t, v) \in V_+ \mid (t - 1/2)^2 + v^2 = 1/4\} = \{(t, v) \in V_+ \mid t(1 - t) = v^2\}.$$

We defined homeomorphisms

$$S'(V_+) \simeq S_+(V_+)/S(V) \simeq S^V \simeq S(V_+) \simeq B(V)/S(V)$$

by letting  $P$ ,  $Q$ ,  $R$ ,  $S$  and  $T$  in the following diagram correspond to each other.



We claimed that the formulae are as follows: Formulae can be read off from the following table:

| $P$                          | $Q$                          | $R$                | $S$                            | $T$                 |
|------------------------------|------------------------------|--------------------|--------------------------------|---------------------|
| $S'(V_+)$                    | $S_+(V_+)/S(V)$              | $S^V$              | $S(V_+)$                       | $B(V)/S(V)$         |
| $(p, u)$                     | $(p, u)/\sqrt{p}$            | $u/p$              | $(2p - 1, 2u)$                 | $u/\sqrt{p}$        |
| $q(q, v)$                    | $(q, v)$                     | $v/q$              | $(2q^2 - 1, 2qv)$              | $v$                 |
| $(1, w)/(1 + w^2)$           | $(1, w)/\sqrt{1 + w^2}$      | $w$                | $(w^2 - 1, 2w)/(1 + w^2)$      | $w/\sqrt{1 + w^2}$  |
| $(1 + s, x)/2$               | $(1 + s, x)/\sqrt{2(1 + s)}$ | $x/(1 + s)$        | $(s, x)$                       | $x/\sqrt{2(1 + s)}$ |
| $(1 - y^2, y\sqrt{1 - y^2})$ | $(\sqrt{1 - y^2}, y)$        | $y/\sqrt{1 - y^2}$ | $(1 - 2y^2, 2y\sqrt{1 - y^2})$ | $y$                 |

I'll do the third line of formulae. We have a point  $R$  corresponding to a vector  $w \in V$  and we need to find formulae for  $P$ ,  $Q$ ,  $S$  and  $T$  in terms of  $w$ . The unmarked point directly above  $R$  is of course  $(1, w)$ . The point  $Q$  lies on the line from the origin to  $(1, w)$  and has norm one, so  $Q = (1, w)/\|(1, w)\| = (1, w)/\sqrt{1 + w^2}$ , as claimed in the second column of the table. The point  $T$  is just the projection of  $Q$  on  $V \subset \mathbb{R} \times V$ , so  $T = w/\sqrt{1 + w^2}$ , as in the fifth column. The point  $P$  clearly has the form  $(\lambda, \lambda w)$  for some  $\lambda > 0$  and it lies in  $S'(V_+)$  so  $\lambda(1 - \lambda) = \lambda^2 w^2$  so  $\lambda = 1/(1 + w^2)$  so  $P = (1, w)/(1 + w^2)$  as in the first column. Finally, consider the point  $S$ . We know that  $P$  lies on the sphere of radius  $1/2$  centred at  $(1/2, 0)$ , so  $2P - (1, 0)$  lies on the unit sphere centred at  $0$ . It is also clear that  $2P - (1, 0) = P + (P - (1, 0))$  lies in the line joining  $(1, 0)$  to  $P$ ; we therefore see from the geometry that  $S = 2P - (1, 0) = (1 - w^2, 2w)/(1 + w^2)$ , as in the fourth column.

**Problem 1.6.** By quoting a suitable general theorem, prove that  $\Delta_1 \times \Delta_2 \times \Delta_3$  is homeomorphic to  $\Delta_6$ .

**Solution:** Both are compact convex sets of dimension 6, and so homeomorphic to the ball  $B^6$ .

**Problem 1.7.** Consider the square  $X = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x, y \leq 1\}$  and the edge  $Y = \{(x, 0) \mid 0 \leq x \leq 1\}$ . Prove that  $X/Y$  is homeomorphic to  $B^2$ .

**Solution:** Put  $Z = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq y \leq 1\}$ . This is compact and convex with nonempty interior in  $\mathbb{R}^2$ , so it is homeomorphic to  $B^2$ . Now define  $f: X \rightarrow Z$  by  $f(x, y) = (xy, y)$ . Then  $f$  sends all of  $Y$  to the origin, and restricts to give a bijection  $X \setminus Y \rightarrow Z \setminus O$  with  $f^{-1}(x, y) = (x/y, y)$ . It therefore induces a continuous bijection  $X/Y \rightarrow Z$ , which is a homeomorphism because  $X/Y$  is compact and  $Z$  is Hausdorff. It follows that  $X/Y \simeq B^2$ .

**Problem 1.8.** Let  $X$  be a space, and  $Y$  a closed subspace, and let  $Z$  be any other space. Construct a continuous bijection  $f: (X/Y) \wedge Z_+ \rightarrow (X \times Z)/(Y \times Z)$ .

(In the cases of interest  $f^{-1}$  will be continuous so that  $f$  is a homeomorphism, but there are technical subtleties around this point.)

**Solution:** Let  $p: X \rightarrow X/Y$  and  $q: X/Y \times Z_+ \rightarrow X/Y \wedge Z_+$  and  $r: X \times Z \rightarrow (X \times Z)/(Y \times Z)$  be the evident quotient maps. Define  $f_0: X \times Z \rightarrow (X/Y) \wedge Z_+$  by  $f_0(x, z) = q(p(x), z)$ , or equivalently  $f_0 = q \circ (p \times 1)$ . The map  $p \times 1$  is continuous by standard properties of the product topology, so  $f_0$  is also continuous. If  $y \in Y$  then  $p(y) = *_X/Y$  and  $q(*_X/Y, z) = *$  for all  $z \in Z$  so  $f_0(y, z) = *$ . This proves that  $f_0(Y \times Z) = *$ , so there is an induced map  $f: (X \times Z)/(Y \times Z) \rightarrow (X/Y) \wedge Z_+$  with  $f \circ r = f_0$ . This is continuous by the universal property of the quotient topology. As sets we have

$$\begin{aligned}
(X \times Z)/(Y \times Z) &= ((X \times Z) \setminus (Y \times Z)) \amalg \{*\} \\
&= ((X \setminus Y) \times Z) \amalg \{*\} \\
X/Y &= (X \setminus Y) \amalg \{*\} \\
(X/Y) \wedge Z_+ &= ((X \setminus Y) \amalg \{*\}) \wedge (Z \amalg \{*\}) \\
&= ((X \setminus Y) \times Z) \amalg \{*\}.
\end{aligned}$$

This shows that  $f$  is a bijection.

**Problem 1.9.** If  $X$  and  $Y$  are finite based sets, with  $|X| = n$  and  $|Y| = m$ , what are  $|X \vee Y|$  and  $|X \wedge Y|$ ?

**Solution:** Put  $X' = X \setminus \{*\}$  and  $Y' = Y \setminus \{*\}$ , so  $|X'| = n - 1$  and  $|Y'| = m - 1$ . We then have  $X \vee Y = X' \amalg Y' \amalg \{*\}$ , so

$$|X \vee Y| = (n - 1) + (m - 1) + 1 = n + m - 1.$$

Similarly, we have  $X \wedge Y = (X' \times Y') \amalg \{*\}$ , so

$$|X \wedge Y| = (n - 1)(m - 1) + 1 = nm - n - m + 2.$$

**Problem 1.10.** Recall that the Cayley map  $f: \mathfrak{u}(2) \rightarrow U(2)$  is given by  $f(\beta) = (1 + \beta/2)(1 - \beta/2)^{-1}$ . Put  $X = U(2) \setminus f(\mathfrak{u}(2))$ . Prove that  $X = \{\alpha \in U(2) \mid -1 \text{ is an eigenvalue of } \alpha\}$ , and thus give a homeomorphism

$$g: (\mathbb{C}P^1 \times S^1)/(\mathbb{C}P^1 \times \{-1\}) \rightarrow X.$$

**Solution:** Put

$$\begin{aligned} V &= \{\alpha \in U(2) \mid -1 \text{ is not an eigenvalue of } \alpha\} \\ &= \{\alpha \in U(2) \mid \alpha + 1 \text{ is invertible}\}. \end{aligned}$$

If  $\alpha \in V$  then we can define  $\beta = 2(\alpha - 1)(\alpha + 1)^{-1}$ , and we find that

$$\begin{aligned} \beta^\dagger &= 2(\alpha^\dagger + 1)^{-1}(\alpha^\dagger - 1) = 2(\alpha^{-1} + 1)^{-1}(\alpha^{-1} - 1) \\ &= 2(1 + \alpha)^{-1}(1 - \alpha) = -\beta, \end{aligned}$$

so  $\beta \in \mathfrak{u}(2)$ . We can also check directly that  $f(\beta) = \alpha$ , so  $\alpha \in \text{image}(f)$ . Conversely, if  $\alpha = f(\beta)$  one checks that  $1 + \alpha = 2(1 - \beta/2)^{-1}$  and so

$$(1 + \alpha)(\frac{1}{2} - \frac{1}{4}\beta) = 1,$$

proving that  $1 + \alpha$  is invertible, and thus that  $\alpha \in V$ . Thus  $f(\mathfrak{u}(2)) = V$ , and so

$$X = \{\alpha \in U(2) \mid -1 \text{ is an eigenvalue of } \alpha\}.$$

Thus,  $\alpha \in X$  iff  $1$  is an eigenvalue of  $-\alpha$ , iff  $-\alpha$  has the form  $z \cdot 1_L \oplus 1_{L^\perp}$  for some  $L \in \mathbb{C}P^1$  and  $z \in S^1$ . We can thus define a surjective map  $g: \mathbb{C}P^1 \times S^1 \rightarrow X$  by  $g(L, z) = -z \cdot 1_L \oplus (-1)_{L^\perp}$ . Note that  $g(L, -1) = -1$  independently of  $L$ , so  $g$  factors through a surjective map  $\bar{g}: (\mathbb{C}P^1 \times S^1)/(\mathbb{C}P^1 \times \{-1\}) \rightarrow X$ . If  $g(L, z) = g(L', z')$  then  $z = \det(g(L, z)) = \det(g(L', z')) = z'$ , so  $z = z'$ . If  $z \neq -1$  then  $L = \ker(g(L, z) + z)$  and  $L' = \ker(g(L', z) + z)$ , but  $g(L, z) = g(L', z)$  so also  $L = L'$ . Using this we see that  $\bar{g}$  is a bijection. It is also continuous, with compact source and Hausdorff target, so it is a homeomorphism.

**Problem 1.11.** Let  $V$  be a finite-dimensional complex vector space equipped with a hermitian inner product. Recall that

$$\begin{aligned} U(V) &= \{\alpha \in \text{End}(V) \mid \alpha^\dagger \alpha = 1\} \\ \mathfrak{u}(V) &= \{\alpha \in \text{End}(V) \mid \alpha^\dagger + \alpha = 0\}. \end{aligned}$$

For any  $\alpha \in \text{End}(V)$ , we put

$$\begin{aligned} \|\alpha\| &= \sup\{\|\alpha(v)\| \mid v \in S(V)\} \\ \exp(\alpha) &= \sum_{k=0}^{\infty} \frac{\alpha^k}{k!} \in \text{End}(V) \end{aligned}$$

We also put

$$W = \{\beta \in \mathfrak{u}(V) \mid \|\beta\| < \pi\}.$$

Prove that  $\exp$  maps  $\mathfrak{u}(V)$  to  $U(V)$ . Prove also that if  $\beta \in W$  then  $\exp(\beta)$  has precisely the same eigenspaces as  $\beta$ , and deduce that  $\exp: W \rightarrow U(V)$  is injective.

When proving that  $U(V)$  is a manifold, the notes use the Cayley transform map

$$f_\alpha(\beta) = (1 + \beta/2)(1 - \beta/2)^{-1}\alpha.$$

We could instead have used the function  $\beta \mapsto \exp(\beta)\alpha$ ; this generalises more easily to other Lie groups, but is more fiddly.

**Solution:** We first claim that for  $\beta \in \mathfrak{u}(V)$  all the eigenvalues are purely imaginary, and that eigenvectors of different eigenvalues are orthogonal. (This is a standard fact of linear algebra.) Indeed, let  $v$  and  $w$  be eigenvectors of eigenvalues  $\lambda$  and  $\mu$ . We then have

$$\lambda \langle v, w \rangle = \langle \lambda v, w \rangle = \langle \beta(v), w \rangle = \langle v, \beta^\dagger(w) \rangle = \langle v, -\beta(w) \rangle = \langle v, -\mu w \rangle = -\bar{\mu} \langle v, w \rangle,$$

so  $(\lambda + \bar{\mu})\langle v, w \rangle = 0$ . We can take  $w = v$  and  $\mu = \lambda$  to get  $(\lambda + \bar{\lambda})\|v\|^2 = 0$ , but eigenvectors are nonzero by definition so  $\|v\|^2 > 0$  so  $\lambda + \bar{\lambda} = 0$ , which means that  $\lambda \in i\mathbb{R}$ . Using this, the more general relation  $(\lambda + \bar{\mu})\langle v, w \rangle = 0$  becomes  $(\lambda - \mu)\langle v, w \rangle = 0$ , which means that  $\langle v, w \rangle = 0$  whenever  $\lambda \neq \mu$ , as claimed.

We next claim that there exists an orthonormal basis for  $V$  consisting of eigenvectors of  $\beta$ . Indeed, we can certainly choose a an eigenvector and then normalise it to get a unit eigenvector  $v_0$  of eigenvalue  $it_0$  say. If  $w$  is perpendicular to  $v_0$  then  $\langle \beta(w), v_0 \rangle = \langle w_0, \beta(v_0) \rangle = -it_0 \langle w_0, v_0 \rangle = 0$ , so  $\beta(w)$  is also perpendicular to  $v_0$ . Thus,  $\beta$  induces an endomorphism of the space  $v_0^\perp$ , which has lower dimension than  $V$ . We can thus assume inductively that there is an orthonormal basis  $v_1, \dots, v_{d-1}$  for  $v_0^\perp$  consisting of eigenvectors of  $\beta$ , of eigenvalues  $it_k$  say. It follows that  $v_0, v_1, \dots, v_{d-1}$  is a basis for  $V$  of the required type. By writing everything in terms of this basis, we can check that  $\|\beta\| = \max\{|t_k| \mid k < d\}$ .

It is now easy to see that  $\beta^m(v_k) = (it_k)^m v_k$  and so  $\exp(\beta)(v_k) = e^{it_k} v_k$  and  $\exp(-\beta)(v_k) = e^{-it_k} v_k$ . From this we see that  $\exp(-\beta)$  is both inverse and adjoint to  $\exp(\beta)$ , so  $\exp(\beta) \in U(V)$ .

Now suppose that  $\|\beta\| < \pi$ , or equivalently  $|t_k| < \pi$  for all  $k$ . As the map  $t \mapsto e^{it}$  is injective on  $(-\pi, \pi)$  we see that  $e^{it_j} = e^{it_k}$  iff  $t_j = t_k$ , and thus the eigenspaces of  $\beta$  are precisely the same as the eigenspaces for  $\exp(\beta)$ . This means that  $\beta$  is the unique map that acts as  $it$  on the  $e^{it}$ -eigenspace of  $\exp(\beta)$ , for all  $t \in (-\pi, \pi)$ . It follows that  $\exp: W \rightarrow U(V)$  is injective.

**Problem 1.12.** Consider the group  $C = \{1, g\}$  with  $g^2 = 1$ . This can be made to act on  $S^2$  in three different ways, listed below. In each case identify the orbit space  $S^2/C$  and decide whether it is a manifold.

- (a)  $g.[x, y, z] = [-x, y, z]$
- (b)  $g.[x, y, z] = [-x, -y, z]$
- (c)  $g.[x, y, z] = [-x, -y, -z]$ .

(Here (b) is significantly harder than (a) or (c). You might want to start by thinking about the analogue of (b) for  $S^1$ , and by considering  $S^1$  as a subspace of  $\mathbb{C}$ .)

**Solution:**

- (a) Here we have a homeomorphism  $f: S^2/C \rightarrow B^2$  given by  $f([x, y, z]) = [y, z]$ , and  $B^2$  is not a manifold. (It is a “manifold with boundary”, but we did not discuss those.)
- (b) If we let  $C$  act on  $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$  by  $g.z = -z$ , then we have a homeomorphism  $f: S^1/C \rightarrow S^1$  given by  $f(z) = z^2$ . In terms of real coordinates, the formula is  $f([x, y]) = [x^2 - y^2, 2xy]$  (because  $(x + iy)^2 = (x^2 - y^2) + 2xyi$ ). Now let  $S_r^1$  be the circle of radius  $r$  around the origin in  $S^1$ . We have a similar homeomorphism  $f: S_r^1/C \rightarrow S_r^1$  given by

$$f([x, y]) = [(x^2 - y^2)/r, 2xy/r] = \left[ \frac{x^2 - y^2}{\sqrt{x^2 + y^2}}, \frac{2xy}{\sqrt{x^2 + y^2}} \right]$$

We can regard  $S^2$  as the union of circles  $S^1_{\sqrt{1-z^2}} \times \{z\}$  for  $-1 \leq z \leq 1$ . Using this we see that there is a homeomorphism  $f: S^2/C \rightarrow S^2$  given by

$$f([x, y, z]) = \left[ \frac{x^2 - y^2}{\sqrt{x^2 + y^2}}, \frac{2xy}{\sqrt{x^2 + y^2}}, z \right].$$

(A little work is needed to check that  $f$  and  $f^{-1}$  are continuous at the north pole and the south pole, but we leave that to the reader.)

- (c) Here the orbit space is, essentially by definition, the real projective space  $\mathbb{R}P^2$ , which is a manifold.

**Problem 1.13.** Which of the following are manifolds?

- (a)  $M_0 = \{(x, y) \in \mathbb{R}^2 \mid xy = 0\}$
- (b)  $M_1 = \{(x, y) \in \mathbb{R}^2 \mid xy < 1\}$
- (c)  $M_2 = \{(x, y) \in \mathbb{R}^2 \mid xy \leq 1\}$
- (d)  $M_3 = \{(x, y) \in \mathbb{R}^2 \mid xy = 1\}$ .

**Solution:**  $M_0$  is not a manifold, because  $(0, 0) \in M_0$  but no neighbourhood of  $(0, 0)$  in  $M_0$  is homeomorphic to  $\mathbb{R}$  (or to  $\mathbb{R}^m$  for any other  $m$ ).

$M_1$  is open in  $\mathbb{R}^2$  and so is a manifold.

$M_2$  is not a manifold, because  $(1, 1) \in M_2$  but no neighbourhood of  $(1, 1)$  in  $M_2$  is homeomorphic to  $\mathbb{R}^m$  for any  $m$ .

$M_3$  is a manifold, homeomorphic to the disjoint union of two copies of  $\mathbb{R}$ .

**Problem 1.14.** Let  $U$  and  $V$  be finite-dimensional vector spaces over  $\mathbb{R}$  equipped with inner products. Suppose that  $\dim(U) = n \geq 2$  and  $\dim(V) = m \geq 1$ . We can think of  $S(U)$  as a subspace of  $S(U \oplus V)$ , and then form the complement  $S(U \oplus V) \setminus S(U)$  or the collapsed space  $S(U \oplus V)/S(U)$ .

- (a) Prove that the inclusion  $i: S(U) \rightarrow S(U \oplus V)$  is homotopic to a constant map. What does this imply about the resulting map  $i^*$  of cohomology rings?
- (b) Prove that  $S(U \oplus V) \setminus S(U)$  is homotopy equivalent to  $S(V)$ .
- (c) Find the cohomology of  $S(U \oplus V)/S(U)$  (paying attention to the special features that occur when  $m = 1$ ).
- (d) In the case where  $V = \mathbb{R}$ , describe the space  $S(U \oplus \mathbb{R})/S(U)$  more explicitly. You may wish to start by drawing pictures for the case where  $U = \mathbb{R}^2$  and  $V = \mathbb{R}$ .

**Solution:**

- (a) It is assumed that  $V \neq 0$ , so we can choose a unit vector  $e \in V$  and define a constant map  $c: S(U) \rightarrow S(U \oplus V)$  by  $c(u) = (0, e)$  for all  $u$ . We can then define  $h: [0, 1] \times S(U) \rightarrow S(U \oplus V)$  by

$$h(t, u) = (\cos(\pi t/2)u, \sin(\pi t/2)e).$$

This is clearly well-defined and continuous, and it lands in  $S(U \oplus V)$  because

$$\|(\cos(\pi t/2)u, \sin(\pi t/2)e)\|^2 = \cos(\pi t/2)^2 \|u\|^2 + \sin(\pi t/2)^2 \|e\|^2 = \cos(\pi t/2)^2 + \sin(\pi t/2)^2 = 1.$$

It satisfies  $h(0, u) = (u, 0) = i(u)$  and  $h(1, u) = (0, e) = c(u)$  so  $i$  is homotopic to  $c$ . It follows that  $i^* = c^*: H^*(S(U \oplus V)) \rightarrow H^*(S(U))$ , and thus that  $i^* = 0$  on  $H^k$  for  $k > 0$ .

- (b) We can define an inclusion  $j: S(V) \rightarrow S(U \oplus V) \setminus S(U)$  by  $j(v) = (0, v)$ . Next, we are identifying  $S(U)$  with  $\{(u, v) \in S(U \oplus V) \mid v = 0\}$ , so

$$S(U \oplus V) \setminus S(U) = \{(u, v) \in S(U \oplus V) \mid v \neq 0\}.$$

For points  $(u, v)$  in this space we have  $v \neq 0$  so  $\|v\| > 0$  so it is legitimate to divide by  $\|v\|$ . We can thus define  $r: S(U \oplus V) \setminus S(U) \rightarrow S(V)$  by  $r(u, v) = v/\|v\|$ . (This does indeed land in  $S(V)$ , because  $\|v/\|v\|\| = \|v\|/\|v\| = 1$ .) It is clear that  $r \circ j = \text{id}: S(V) \rightarrow S(V)$ . In the opposite direction, we can define

$$h: [0, 1] \times (S(U \oplus V) \setminus S(U)) \rightarrow S(U \oplus V) \setminus S(U)$$

by  $h(t, (u, v)) = (tu, v)/\|(tu, v)\|$ . To see that this is valid:

- We are assuming that  $(u, v) \in S(U \oplus V) \setminus S(U)$ , so  $v \neq 0$ . This means that  $(tu, v) \neq 0$  so  $\|(tu, v)\| > 0$  so it is legitimate to divide by  $\|(tu, v)\|$ .
- As  $\|(tu, v)/\|(tu, v)\|\| = 1$  we have  $h(t, (u, v)) \in S(U \oplus V)$
- The second component in  $h(t, (u, v))$  is a nonzero multiple of  $v$ , and  $v \neq 0$ , so the second component is nonzero, so  $h(t, (u, v)) \in S(U \oplus V) \setminus S(U)$ .

We have  $h(0, (u, v)) = (0, v/\|v\|) = j(r(u, v))$  and  $h(1, (u, v)) = (u, v)$  so  $j \circ r$  is homotopic to the identity. This proves that  $S(U \oplus V) \setminus S(U)$  is homotopy equivalent to  $S(V)$ .

- (c) Put  $Q = S(U \oplus V)/S(U)$  for brevity. As  $n, m > 0$  we see that  $S(U)$  and  $S(U \oplus V)$  are both path connected and it follows that  $Q$  is also path connected. We therefore have  $\tilde{H}^0(Q) = 0$ . For  $k > 0$  we have an exact sequence

$$H^{k-1}(S(U \oplus V)) \xrightarrow{i^*} H^{k-1}(S(U)) \xrightarrow{\delta} \tilde{H}^k(Q) \xrightarrow{\alpha} H^k(S(U \oplus V)) \xrightarrow{i^*} H^k(S(U)).$$

In the case where  $k = 1$  this takes the form

$$\mathbb{Z} \xrightarrow{\text{id}} \mathbb{Z} \xrightarrow{\delta} \tilde{H}^1(Q) \xrightarrow{\alpha} 0 \xrightarrow{i^*} H^1(S(U)).$$

(We have filled in  $H^1(S(U \oplus V)) = 0$  which is valid as  $\dim(U \oplus V) = n + m \geq 3$ .) From this it is clear that  $\tilde{H}^1(Q) = 0$ . Now suppose that  $k > 1$ , so  $k, k - 1 > 0$ . As  $i$  is homotopic to a constant map it acts as zero on cohomology in strictly positive degrees, so both of the maps marked  $i^*$  in our sequence are zero, which means that we actually have a short exact sequence

$$0 \rightarrow H^{k-1}(S(U)) \xrightarrow{\delta} \tilde{H}^k(Q) \xrightarrow{\alpha} H^k(S(U \oplus V)) \rightarrow 0.$$

The group  $H^{k-1}(S(U))$  is only nonzero if  $k = n$ , and the group  $H^k(S(U \oplus V))$  is only nonzero if  $k = n + m - 1$ . Thus, for  $k > 0$  with  $k \notin \{n, n + m - 1\}$  we have  $\tilde{H}^k(Q) = 0$ . If  $m > 1$  then  $n$  and  $n + m - 1$  are different and we find that  $\tilde{H}^n(Q) = \tilde{H}^{n+m-1}(Q) = \mathbb{Z}$ . As always the reduced and unreduced cohomology groups are the same in strictly positive degrees, and we conclude that

$$H^i(S(U \oplus V)/S(U)) = H^i(Q) = \begin{cases} \mathbb{Z} & \text{if } i \in \{0, n, n + m - 1\} \\ 0 & \text{otherwise} \end{cases}$$

In the case  $m = 1$ , we have a short exact sequence

$$0 \rightarrow \mathbb{Z} = H^{n-1}(S(U)) \xrightarrow{\delta} \tilde{H}^n(Q) \xrightarrow{\alpha} H^n(S(U \oplus V)) = \mathbb{Z} \rightarrow 0$$

If we put  $a = \delta(1)$  and choose any  $b$  with  $\alpha(b) = 1$  then it is not hard to deduce that  $\tilde{H}^n(Q) = \mathbb{Z}\{a, b\} \simeq \mathbb{Z}^2$ . We thus have

$$H^i(S(U \oplus V)/S(U)) = H^i(Q) = \begin{cases} \mathbb{Z} & \text{if } i = 0 \\ \mathbb{Z}^2 & \text{if } i = n \\ 0 & \text{otherwise} \end{cases}$$

- (d) In the case where  $m = 2$  the space  $S(U \oplus \mathbb{R})$  is just the ordinary sphere  $S^2$  and the subspace  $S(U)$  is the equator. If we collapse the equator to a point then each of the two hemispheres becomes a smaller sphere so we are left with the union of two spheres joined at a point, which is denoted by  $S^2 \vee S^2$ . Essentially the same thing happens for arbitrary  $m$ , as we now explain. We have

$$S(U \oplus \mathbb{R}) = \{(u, t) \in U \oplus \mathbb{R} \mid \|u\|^2 + t^2 = 1\}$$

$$C = S(U \oplus \mathbb{R}) \setminus S(U) = \{(u, t) \in U \oplus \mathbb{R} \mid \|u\|^2 + t^2 = 1, t \neq 0\}.$$

We can write  $C$  as the disjoint union of the sets  $C_+ = \{(u, t) \in C \mid t > 0\}$  and  $C_- = \{(u, t) \in C \mid t < 0\}$ . We have a homeomorphism from  $C_+$  to the open unit ball in  $U$  given by  $(u, t) \mapsto u$ , and the open unit ball is homeomorphic to the full space  $U$  by  $u \mapsto u/\sqrt{1 - \|u\|^2}$ , and  $U \cup \{\infty\}$  is homeomorphic to  $S(U \oplus \mathbb{R})$  by the stereographic projection map  $u \mapsto (2u, \|u\|^2 - 1)/(\|u\|^2 + 1)$ . By combining these homeomorphisms we get  $C_+ \cup \{\infty\} \simeq S(U \oplus \mathbb{R})$ , and similarly  $C_- \cup \{\infty\} \simeq S(U \oplus \mathbb{R})$ , so  $Q$  is homeomorphic to two copies of  $S(U \oplus \mathbb{R})$  joined at a point. One can also obtain the more direct formula  $(u, t) \mapsto (2tu, 2t^2 - 1)$  for the homeomorphism  $C_+ \cup \{\infty\} \rightarrow S(U \oplus \mathbb{R})$ . (With slightly different notation, this appears in the table on page 15 of `examples.pdf`.)

## 2. HOMOTOPY

**Problem 2.1.** Recall that  $X$  is a *homotopy retract* of  $Y$  if there exist continuous maps  $X \xrightarrow{i} Y \xrightarrow{r} X$  such that the composite  $ri: X \rightarrow X$  is homotopic to the identity. By considering the effect of such maps on cohomology rings, show that the sphere  $S^4$  is not a homotopy retract of the 4-torus  $T = S^1 \times S^1 \times S^1 \times S^1$ .

**Solution:** We have  $H^*S^4 = \mathbb{Z}[u]/u^2$  with  $u \in H^4S^4$ , and  $H^*S^1 = E[x] = \mathbb{Z}\{1, x\}$  with  $x \in H^1S^1$ . Using the Künneth theorem we deduce that

$$H^*T = (H^*S^1) \otimes (H^*S^1) \otimes (H^*S^1) \otimes (H^*S^1) = E[x_0, x_1, x_2, x_3],$$

an exterior algebra with 4 generators  $x_i$  in  $H^1T$ . In particular,  $H^4T = \mathbb{Z}.x_0x_1x_2x_3$ .

Now suppose (for a contradiction) we have maps  $S^4 \xrightarrow{i} T \xrightarrow{r} S^4$  such that  $ri: S^4 \rightarrow S^4$  is homotopic to the identity. It follows that the ring map  $i^*r^* = (ri)^*: H^*S^4 \rightarrow H^*S^4$  is the identity, so  $i^*(r^*(u)) = u$ . Now  $r^*(u)$  lies in the group  $H^4T = \mathbb{Z}.x_0x_1x_2x_3$ , so  $r^*(u) = m.x_0x_1x_2x_3$  for some  $m$ , so  $i^*(r^*(u)) = m.i^*(x_0)i^*(x_1)i^*(x_2)i^*(x_3)$ . However,  $i^*(x_0)$  lies in the group  $H^1S^4 = 0$ , so  $i^*(x_0) = 0$ , so  $i^*(r^*(u)) = 0$ . This contradicts the fact that  $i^*(r^*(u)) = u$ . Thus, no such maps  $i$  and  $r$  can exist.

**Problem 2.2.** Recall that  $X$  is a *homotopy retract* of  $Y$  if there exist continuous maps  $X \xrightarrow{i} Y \xrightarrow{r} X$  such that the composite  $ri: X \rightarrow X$  is homotopic to the identity.

Let  $U(n)$  be the group of  $n \times n$  unitary matrices, and let  $T(m)$  be the  $m$ -torus (i.e. the product of  $m$  copies of  $S^1$ ). By considering the relevant cohomology rings, prove that if  $n, m > 1$  then  $U(n)$  is not a homotopy retract of  $T(m)$  and also  $T(m)$  is not a homotopy retract of  $U(n)$ .

**Solution:** Any map  $f: U(n) \rightarrow T(m)$  gives a ring map

$$f^*: H^*(T(m)) = E[x_1, \dots, x_m] \rightarrow H^*(U(n)) = E[a_1, a_3, \dots, a_{2n-1}].$$

Here  $|x_i| = 1$  and  $|a_{2j+1}| = 2j + 1$ . In particular, in degree one we have a homomorphism

$$f^*: H^1(T(m)) = \mathbb{Z}\{x_1, \dots, x_m\} \simeq \mathbb{Z}^m \rightarrow H^1(U(n)) = \mathbb{Z}.a_1,$$

so  $f^*(x_i) = p_i a_1$  for some  $p_i \in \mathbb{Z}$ . As  $a_1^2 = 0$  it follows that  $f^*(x_i x_j) = 0$  and thus that  $f^* = 0$  in degrees bigger than one. It follows that  $f$  cannot be surjective or injective.

If  $U(n)$  was a homotopy retract of  $T(m)$  we would have maps  $U(n) \xrightarrow{f} T(m) \xrightarrow{g} U(n)$  with  $gf \simeq 1$  so  $f^*g^* = 1$  so  $f^*$  would be surjective, which is impossible. If  $T(m)$  was a homotopy retract of  $U(n)$  we would have maps  $T(m) \xrightarrow{g} U(n) \xrightarrow{f} T(m)$  with  $fg \simeq 1$  so  $g^*f^* = 1$  so  $f^*$  would be injective, which is again impossible.

**Problem 2.3.** Prove that any homotopy equivalence  $f: X \rightarrow Y$  gives a bijection  $\pi_0(X) \simeq \pi_0(Y)$  (where  $\pi_0(X)$  is the set of path-components of  $X$ ). In particular, check that if  $X$  is path-connected and  $Y$  is homotopy equivalent to  $X$  then  $Y$  is path-connected.

**Solution:** This is an immediate consequence of the fact that  $\pi_0$  is a homotopy invariant functor. We can spell this out as follows.

- (a) First, let  $f: X \rightarrow Y$  be any continuous map. If  $[x_0] = [x_1]$  in  $\pi_0(X)$  then there is a continuous map  $u: [0, 1] \rightarrow X$  with  $u(0) = x_0$  and  $u(1) = x_1$ . This means that the composite  $f \circ u: [0, 1] \rightarrow Y$  is a path from  $f(x_0)$  to  $f(x_1)$  in  $Y$ , so  $[f(x_0)] = [f(x_1)]$  in  $\pi_0(Y)$ . This means that we have a well-defined map  $f_*: \pi_0(X) \rightarrow \pi_0(Y)$  given by  $f_*([x]) = [f(x)]$  for all  $x \in X$ .

- (b) If we have continuous maps  $X \xrightarrow{f} Y \xrightarrow{g} Z$  then we have

$$(g \circ f)_*([x]) = [(g \circ f)(x)] = [g(f(x))] = g_*([f(x)]) = g_*(f_*([x])) = (g_* \circ f_*)([x]).$$

This shows that  $(g \circ f)_* = g_* \circ f_*$ , and it is also clear that the identity map  $\text{id}: X \rightarrow X$  has  $\text{id}_* = \text{id}: \pi_0(X) \rightarrow \pi_0(X)$ , so we have defined a functor  $\pi_0$  from spaces to sets.

- (c) Now suppose we have maps  $f, g: X \rightarrow Y$ , and that  $f$  and  $g$  are homotopic. This means that we can choose a homotopy  $h: [0, 1] \times X \rightarrow Y$  with  $h(0, x) = f(x)$  and  $h(1, x) = g(x)$  for all  $x \in X$ . Thus, for any fixed  $x$  we can define  $v_x: [0, 1] \rightarrow Y$  by  $v_x(t) = h(t, x)$ , and this is a path from  $f(x)$  to  $g(x)$ , proving that  $[f(x)] = [g(x)]$  in  $\pi_0(Y)$ . In other words, we have  $f_*([x]) = g_*([x])$  for all  $x$ , so  $f_* = g_*: \pi_0(X) \rightarrow \pi_0(Y)$ .
- (d) Now suppose that  $f: X \rightarrow Y$  is a homotopy equivalence, so we can choose  $g: Y \rightarrow X$  such that  $g \circ f$  is homotopic to  $\text{id}_X$  and  $f \circ g$  is homotopic to  $\text{id}_Y$ . Using (b) and (c) we have

$$g_* \circ f_* = (g \circ f)_* = (\text{id}_X)_* = \text{id}_{\pi_0(X)}$$

$$f_* \circ g_* = (f \circ g)_* = (\text{id}_Y)_* = \text{id}_{\pi_0(Y)}.$$

This proves that the maps  $f_*: \pi_0(X) \rightarrow \pi_0(Y)$  and  $g_*: \pi_0(Y) \rightarrow \pi_0(X)$  are inverse to each other, so both are bijections.

- (e) In particular we have  $|\pi_0(X)| = 1$  iff  $|\pi_0(Y)| = 1$ , so  $X$  is path-connected iff  $Y$  is path-connected.

**Problem 2.4.** Let  $X$  be the set of continuous maps  $f: S^1 \rightarrow S^1$ . Define a metric and thus a topology on  $X$ . Using winding numbers and the theory of coverings, give a homeomorphism  $X \simeq \mathbb{Z} \times S^1 \times Y$ , where  $Y = \{g: S^1 \rightarrow \mathbb{R} \mid g(1) = 0\}$ . Describe  $H^*X$ .

**Solution:** We define a metric on  $X$  by  $d(f_0, f_1) = \sup_{z \in S^1} \|f_0(z) - f_1(z)\|$ . Define a map  $e: \mathbb{R} \rightarrow S^1$  by  $e(t) = e^{2\pi it}$ , and define  $\phi: \mathbb{Z} \times S^1 \times Y \rightarrow X$  by  $\phi(n, w, g)(z) = z^n w e(g(z))$ . In the opposite direction, we define  $\psi_0: X \rightarrow \mathbb{Z}$  by  $\psi_0(f) = \text{winding number of } f(S^1) \text{ around } 0$ . If  $d(f_0, f_1) < 2$  then for all  $z$  we see that 0 does not lie on the line segment from  $f_0(z)$  to  $f_1(z)$ . We can thus define  $h_t(z) = t f_1(z) + (1-t)f_0(z)$  and  $f_t(z) = h_t(z)/|h_t(z)|$  to get a homotopy from  $f_0$  to  $f_1$ , so that  $\psi_0(f_0) = \psi_0(f_1)$ . This shows that  $\psi_0$  is locally constant and thus continuous. Next, define  $\psi_1: X \rightarrow S^1$  by  $\psi_1(f) = f(1)$ . Clearly  $|\psi_1(f_0) - \psi_1(f_1)| \leq d(f_0, f_1)$  and thus  $\psi_1$  is continuous. Now write

$$Z = \{f \in X \mid \psi_0(f) = 0 \text{ and } \psi_1(f) = 1\}.$$

Define  $\chi: X \rightarrow Z$  by  $\chi(f)(z) = z^{-n} f(1)^{-1} f(z)$ , and define  $\xi: Y \rightarrow Z$  by  $\xi(g)(z) = e(g(z))$ . We next claim that  $\xi$  is a homeomorphism. To see this, we need some theory of covering spaces. Let  $U, V$  and  $W$  be spaces with chosen basepoints  $u_0, v_0$  and  $w_0$ . Let  $f: U \rightarrow V$  be a continuous map, let  $q: W \rightarrow V$  be a covering map, and suppose that  $f(u_0) = v_0 = q(w_0)$ . Suppose also that the map  $f_*: \pi_1(U, u_0) \rightarrow \pi_1(V, v_0)$  is trivial. A standard result then says that there is a unique map  $g: U \rightarrow W$  with  $g(u_0) = w_0$  and  $qg = f$ . We now take  $U = V = S^1$  and  $W = \mathbb{R}$ , with basepoints  $1 \in S^1$  and  $0 \in \mathbb{R}$ . We take  $q$  to be  $e: \mathbb{R} \rightarrow S^1$  and we let  $f$  be any point in  $Z$ . Because  $\psi_1(f) = 1$  we know that  $f$  preserves basepoints, and because  $\psi_0(f) = 0$  we know that  $\pi_1(f)$  is trivial. It follows that there is a unique  $g \in Y$  with  $f = \xi(g)$ , so  $\xi$  is a bijection. To check that  $\xi^{-1}$  is continuous, fix  $f_0 \in Y$  and put  $g_0 = \xi^{-1}(f_0)$ . Put  $U = \{f \in Y \mid d(f, f_0) < 2\}$ , and let  $\log(z)$  denote the usual branch of the logarithm that is holomorphic away from the negative real axis. If  $f \in U$  then  $f(z)/f_0(z) \in S^1 \setminus \{-1\}$  for all  $z$ , so the map  $z \mapsto \log(f(z)/f_0(z))$  is continuous. One checks that

$$\xi^{-1}(f)(z) = g_0(z) + \log(f(z)/f_0(z))/2\pi i,$$

and thus that  $\xi^{-1}$  is continuous on the neighbourhood  $U$  of  $f_0$ . As  $f_0$  was arbitrary, we deduce that  $\xi$  is a homeomorphism as claimed. We can thus define  $\psi_2 = \xi^{-1}\chi: X \rightarrow Y$ . It is now easy to check that  $\phi$  and  $\psi = (\psi_0, \psi_1, \psi_2)$  are mutually inverse homeomorphisms between  $X$  and  $\mathbb{Z} \times S^1 \times Y$ . As  $Y$  is a vector space, it is contractible by an obvious linear homotopy, so  $X$  is homotopy equivalent to  $\mathbb{Z} \times S^1 = \coprod_{n \in \mathbb{Z}} S^1$ . It follows that  $H^*X = \prod_{n \in \mathbb{Z}} H^*S^1 = \prod_n \mathbb{Z}[u]/u^2$ .

**Problem 2.5.** Recall that

$$\begin{aligned} \mathbb{R}^\infty &= \{x \in \prod_{k=0}^{\infty} \mathbb{R} \mid x_k = 0 \text{ for } k \gg 0\} \\ &= \bigoplus_k \mathbb{R} = \bigcup_k \mathbb{R}^k, \end{aligned}$$

with the colimit topology.

$$S^\infty = \{x \in \mathbb{R}^\infty \mid \sum_k x_k^2 = 1\} = \bigcup_k S^k.$$

Verify that the topology on  $S^\infty$  as a subspace of  $\mathbb{R}^\infty$  is the same as its topology as the colimit of the spaces  $S^k$ . Prove that  $S^\infty$  is contractible. You may like to consider the maps  $\mu(a, b, c): \mathbb{R}[t] \rightarrow \mathbb{R}[t]$  given by  $f \mapsto a + (b + ct)f$ .

**Solution:** Let  $\tau$  be the colimit topology on  $S^\infty$ , let  $\tau'$  be the colimit topology on  $\mathbb{R}^\infty$ , and let  $\tau''$  be the subspace topology obtained by restricting  $\tau'$  to  $S^\infty$ . As  $S^\infty \cap \mathbb{R}^n = S^{n-1}$  is closed in the usual topology on  $\mathbb{R}^n$  for all  $n$ , we see that  $S^\infty$  is  $\tau'$ -closed in  $\mathbb{R}^\infty$ . It follows that a subset  $X \subseteq S^\infty$  is  $\tau''$ -closed in  $S^\infty$  iff it is  $\tau'$ -closed in  $\mathbb{R}^\infty$ . By the definition of  $\tau'$ , this holds iff  $X \cap \mathbb{R}^n$  is closed in  $\mathbb{R}^n$  for all  $n$ . As  $X \subseteq S^\infty$  we see that  $X \cap \mathbb{R}^n = X \cap S^{n-1}$ , and by ordinary finite-dimensional topology it is clear that this set is closed in  $S^{n-1}$  iff it is closed in  $\mathbb{R}^n$ . On the other hand, we see from the definition of  $\tau$  that  $X$  is  $\tau$ -closed iff  $X \cap S^{n-1}$  is closed in  $S^{n-1}$  for all  $n$ ; putting this all together, we see that  $X$  is  $\tau$ -closed iff it is  $\tau''$ -closed, so  $\tau = \tau''$  as claimed.

We next show that  $S^\infty$  is contractible. The map  $(t, u) \mapsto tu$  gives a homeomorphism  $\mathbb{R}^+ \times S^\infty \rightarrow \mathbb{R}^\infty \setminus \{0\}$  and thus a homotopy equivalence  $S^\infty \simeq \mathbb{R}^\infty \setminus \{0\}$ . We can identify  $\mathbb{R}^\infty$  with  $\mathbb{R}[t]$ , so it suffices to show that  $\mathbb{R}[t] \setminus \{0\}$  is contractible. Consider the map  $\mu(a, b, c): \mathbb{R}[t] \rightarrow \mathbb{R}[t]$  defined by  $\mu(a, b, c)(f) = a + (b + ct)f$ . Put

$$M = \{(a, b, c) \neq (0, 0, 0) \mid c \neq 0 \text{ or } a = 0 \text{ or } b = 0\}.$$

I claim that if  $(a, b, c) \in M$  and  $a + (b + ct)f = 0$  then  $f = 0$ . Indeed, suppose not. The polynomial  $(b + ct)f = -a$  is constant and  $f \neq 0$  so it is clear that  $c = 0$  and  $f$  is a nonzero constant with  $bf = -a$ . As  $(a, b, c) \in M$  and  $c = 0$  we have  $a = 0$  or  $b = 0$  and thus the equation  $bf = -a$  gives  $a = b = c = 0$ , contradicting the assumption  $(a, b, c) \neq (0, 0, 0)$  in the definition of  $M$ . Thus, for  $(a, b, c) \in M$  we have a map  $\mu(a, b, c): \mathbb{R}[t] \setminus \{0\} \rightarrow \mathbb{R}[t] \setminus \{0\}$ . Next note that  $\mu(0, 1, 0)$  is the identity and  $\mu(1, 0, 0)$  is the constant map with value 1 so it suffices to join  $(1, 0, 0)$  to  $(0, 1, 0)$  by a path in  $M$ . We can do this by moving in straight lines from  $(1, 0, 0)$  to  $(1, 0, 1)$ , then to  $(0, 0, 1)$ , then to  $(0, 1, 0)$ .

### 3. COCHAIN COMPLEXES

**Problem 3.1.** Define a cochain complex  $U^*$  as follows. For all  $n \in \mathbb{Z}$ , the group  $U^n$  is generated by an element  $u_n$  with  $100u_n = 0$ , so  $U^n \simeq \mathbb{Z}/100$ . The differential  $d: U^n \rightarrow U^{n+1}$  is given by  $d(u_n) = 10u_{n+1}$ . Prove that  $H^*U = 0$  but that the identity map  $1: U^* \rightarrow U^*$  is not cochain-homotopic to zero.

**Solution:** First, we have

$$\begin{aligned} Z^n U &= \{z \in U^n \mid d(z) = 0\} = \{mu_n \mid m \in \mathbb{Z}/100, 10mu_{n+1} = 0\} \\ &= \{mu_n \mid m \in \mathbb{Z}/100, 10m = 0 \pmod{100}\} \\ &= \{10ku_n \mid k \in \mathbb{Z}/100\} \\ &= \{d(ku_{n-1}) \mid k \in \mathbb{Z}/100\} = B^n U \\ H^n U &= (Z^n U)/(B^n U) = 0. \end{aligned}$$

Now suppose (for a contradiction) that the identity map  $1: U \rightarrow U$  is cochain-homotopic to zero, so there exists a sequence of maps  $s_n: U^n \rightarrow U^{n-1}$  with  $ds + sd = 1$ . This must have  $s_n(u_n) = m_n u_{n-1}$  for some  $m_n \in \mathbb{Z}/100$ . The relation  $(ds + sd)(u_n) = u_n$  then becomes  $10(m_{n-1} + m_n)u_n = u_n$ , and we can multiply by 10 to get  $0 = 10u_n$ , which is false. Thus, no such cochain homotopy can exist.

**Problem 3.2.** Let  $U^* \xrightarrow{i} V^* \xrightarrow{p} W^*$  be a short exact sequence of cochain complexes and cochain maps. Suppose that all the groups  $H^k(U)$ ,  $H^k(V)$  and  $H^k(W)$  are finite, and that  $|H^j(U)|$  and  $|H^k(V)|$  are coprime for all  $j$  and  $k$ . What can you deduce about  $H^*(W)$ ? (If you make any general claims about the theory of finite abelian groups, you should give at least a brief justification.)

**Solution:** We first claim that if  $A$  and  $B$  are finite abelian groups of coprime order then every homomorphism  $\phi: A \rightarrow B$  is zero. Indeed, we can choose integers  $x, y$  with  $x|A| + y|B| = 1$ . Every element of  $A$  has order divisible by  $|A|$  so  $|A|\phi = 0$ . Every element of  $B$  has order divisible by  $|B|$  so  $|B|\phi = 0$ . It follows that  $(x|A| + y|B|)\phi = 0$  so  $\phi = 0$ .

Next, we claim that if  $A \xrightarrow{j} B \xrightarrow{q} C$  is a short exact sequence of finite abelian groups with  $|A|$  and  $|C|$  coprime, then  $B \simeq A \oplus C$ . Indeed, we can certainly write  $B$  as a direct sum of cyclic groups of prime power order, and each of the relevant primes must divide  $|A|$  or  $|C|$  but not both. Let  $A'$  be the sum of the groups where the relevant prime divides  $|A|$ , and let  $C'$  be the sum of the groups where the relevant prime divides  $|C|$ . Then  $|A'||C'| = |B| = |A||B|$  and by considering the prime factorisations we deduce that  $|A'| = |A|$  and  $|C'| = |C|$ . It then follows easily that  $j$  gives an isomorphism  $A \rightarrow A'$  and  $q$  gives an isomorphism  $C' \rightarrow C$ .

Now suppose we have a short exact sequence  $U^* \xrightarrow{i} V^* \xrightarrow{p} W^*$  as in the question. The Snake Lemma gives exact sequences

$$H^k(U) \xrightarrow{i_*} H^k(V) \xrightarrow{p_*} H^k(W) \xrightarrow{\delta} H^{k+1}(U) \xrightarrow{i_*} H^{k+1}(V).$$

The maps  $i_*$  have domain and codomain of coprime order, so they are zero. This means that the middle portion of the above sequence is short exact. The groups  $H^k(V)$  and  $H^{k+1}(U)$  also have coprime order, so we deduce that  $H^k(W) \simeq H^k(V) \oplus H^{k+1}(U)$ .

#### 4. MAYER-VIETORIS

**Problem 4.1.** Say that a subset  $V \subseteq \mathbb{R}^n$  is *subconvex* if it is either convex or empty (so that the intersection of any two subconvex sets is subconvex). Let  $U$  be an open subset of  $\mathbb{R}^n$ , and suppose that  $U = U_1 \cup \dots \cup U_m$  where each set  $U_i$  is open and subconvex. Prove that  $H^j(U) = 0$  for  $j \geq m$ .

[Hint: you will need to use induction, but you should formulate the induction statement carefully.]

**Solution:** We will prove by induction on  $m$  that if  $X \subseteq \mathbb{R}^n$  is the union of  $m$  subconvex open sets then  $H^{\geq m}(X) = 0$ . The base case  $m = 1$  is clear, because then  $X$  itself is either empty (with  $H^*(X) = 0$ ) or convex and therefore contractible (with  $H^*(X) = \mathbb{Z}$ , concentrated in degree zero).

Now suppose we have subconvex open sets  $U_1, \dots, U_{m+1}$  with  $m \geq 1$ . Put  $Y = U_1 \cup \dots \cup U_m$  and  $Z = U_{m+1}$  so  $Y \cup Z = U_1 \cup \dots \cup U_{m+1}$ . Suppose  $j \geq m + 1$ ; we must show that  $H^j(Y \cup Z) = 0$ . We have a Mayer-Vietoris exact sequence

$$H^{j-1}(Y \cap Z) \xrightarrow{\delta} H^j(Y \cup Z) \xrightarrow{\alpha} H^j(Y) \times H^j(Z).$$

We can express  $Y \cap Z$  as the union of the sets  $U_1 \cap U_{n+1}, \dots, U_n \cap U_{n+1}$ , which are again open and subconvex. We can therefore assume inductively that  $H^{j-1}(Y \cap Z) = 0$ . We can also apply the induction hypothesis to the sets  $U_1, \dots, U_n$  to see that  $H^j(Y) = 0$ . As  $Z$  is empty/contractible and  $j \geq m + 1 \geq 1$  we also have  $H^j(Z) = 0$ . Exactness of the above sequence therefore gives  $H^j(Y \cup Z) = 0$  as required.

**Problem 4.2.** Put  $A = \{0, 1, \dots, n-1\} \subseteq \mathbb{R}$  and  $U = \mathbb{R}^2 \setminus (A \times \{0\})$ . Calculate  $H^*(U)$ . (Hint: consider the sets  $U_{\pm} = \mathbb{R}^2 \setminus (A \times [0, \pm\infty))$  and use the Mayer-Vietoris sequence.)

**Solution:** Define  $f_+, g_+ : U_+ \rightarrow U_+$  by  $f_+(x, y) = (x, -1)$  and  $g_+(x, y) = (0, -1)$ . We find that the line segment joining  $(x, y)$  to  $f_+(x, y)$  lies wholly in  $U_+$ , as does the line segment joining  $f_+(x, y)$  to  $g_+(x, y)$ . It follows that the identity is homotopic to  $f_+$  by a linear homotopy, and  $f_+$  is homotopic to  $g_+$  by a linear homotopy, so  $U_+$  is contractible. Essentially the same argument shows that  $U_-$  is contractible. On the other hand, we have  $U_+ \cup U_- = U$ , and the intersection is

$$U_+ \cap U_- = \mathbb{R}^2 \setminus (A \times \mathbb{R}) = (\mathbb{R} \setminus \{0, \dots, n-1\}) \times \mathbb{R}.$$

This is the disjoint union of  $n+1$  contractible sets, so  $H^0(U_+ \cap U_-) \simeq \mathbb{Z}^{n+1}$  and  $H^i(U_+ \cap U_-) = 0$  for  $i > 0$ .

It is clear that  $U$  is path connected so  $H^0(U) = \mathbb{Z}$ . For  $i > 1$  we have a Mayer-Vietoris exact sequence

$$0 = H^{i-1}(U_+ \cap U_-) \xrightarrow{\delta} H^i(U) \rightarrow H^i(U_+) \times H^i(U_-) = 0,$$

which gives  $H^i(U) = 0$ . Finally, we also have an exact sequence

$$H^0(U) \xrightarrow{\alpha} H^0(U_+) \times H^0(U_-) \xrightarrow{\beta} H^0(U_+ \cap U_-) \xrightarrow{\delta} H^1(U) \rightarrow H^1(U_+) \times H^1(U_-).$$

This has the form

$$\mathbb{Z} \xrightarrow{\alpha} \mathbb{Z} \times \mathbb{Z} \xrightarrow{\beta} \mathbb{Z}^{n+1} \xrightarrow{\delta} H^1(U) \rightarrow 0,$$

with  $\alpha(n) = (n, n)$  and  $\beta(n, m) = (n-m). (1, \dots, 1)$ . It follows that  $\delta$  gives an isomorphism from  $\mathbb{Z}^{n+1} / \text{image}(\beta)$  to  $H^1(U)$ , and  $\mathbb{Z}^{n+1} / \text{image}(\beta) \simeq \mathbb{Z}^n$  so  $H^1(U) \simeq \mathbb{Z}^n$ .

More precisely, for  $0 \leq k < n$  we can define maps

$$S^1 \xrightarrow{i_k} U \xrightarrow{r_k} S^1$$

by  $i_k(u) = (k, 0) + u/2$  and  $r_k(v) = (v - (k, 0)) / \|v - (k, 0)\|$ . We find that  $r_k \circ i_k$  is the identity, but  $r_k \circ i_j$  is constant if  $j \neq k$ .

We now let  $u_1$  denote the usual generator of  $H^1(S^1)$ , and put  $x_k = r_k^*(u_1) \in H^1(U)$ . We then find that  $i_k^*(x_k) = u_1$ , but  $i_j^*(x_k) = 0$  for  $j \neq k$ , which implies that the elements  $x_k$  generate a subgroup of  $H^1(U)$  that is isomorphic to  $\mathbb{Z}^n$ . A slight elaboration of the Mayer-Vietoris argument proves that the elements  $x_1, \dots, x_n$  form a basis for  $H^1(U)$  over  $\mathbb{Z}$ .

## 5. THE KÜNNETH THEOREM

**Problem 5.1.** Suppose we have a space  $X$  with  $H^0 X \simeq \mathbb{Z}$  and  $H^1(X) \simeq \mathbb{Z}/2$  and  $H^2(X) \simeq \mathbb{Z}/3$ . What can we say about  $H^2(X \times \mathbb{C}P^1)$ ?

**Solution:** Let  $a$  and  $b$  be the generators of  $H^1(X)$  and  $H^2(X)$ , so  $2a = 0$  and  $3b = 0$ . Then the Künneth theorem tells us that  $H^*(X \times \mathbb{C}P^1)$  is generated by  $\{1, a, b, x, ax, bx\}$ , with  $|a| = 1$  and  $|b| = |x| = 2$  and  $2a = 0$ ,  $2ax = 0$ ,  $3b = 0$  and  $3bx = 0$ . It follows that  $H^2(X \times \mathbb{C}P^1)$  is generated by  $x$  and  $b$  subject to  $3b = 0$ , so  $H^2(X \times \mathbb{C}P^1) = \mathbb{Z} \oplus \mathbb{Z}/3$ .

**Problem 5.2.** Consider the spaces  $X = \mathbb{C} \setminus \{0, 1\}$  and  $Y = \mathbb{C} \setminus \{0, 1, 2\}$ . The cohomology of these was described in lectures. Describe  $H^n(X \times Y)$  for all  $n$ . Show that  $a^2 = 0$  for all  $a \in H^1(X \times Y)$ .

**Solution:** We saw in lectures that  $H^*X = \mathbb{Z}\{1, v_0, v_1\}$  and  $H^*Y = \mathbb{Z}\{1, w_0, w_1, w_2\}$ , with  $|v_i| = |w_j| = 1$  and  $v_i v_j = 0 = w_k w_l$ . The Künneth theorem therefore gives

$$\begin{aligned} H^0(X \times Y) &= \mathbb{Z} \\ H^1(X \times Y) &= \mathbb{Z}\{v_0, v_1, w_0, w_1, w_2\} \\ H^2(X \times Y) &= \mathbb{Z}\{v_0 w_0, v_0 w_1, v_0 w_2, v_1 w_0, v_1 w_1, v_1 w_2\}, \end{aligned}$$

and  $H^k(X \times Y) = 0$  for  $k > 2$ . Now suppose we have  $a \in H^1(X \times Y)$ . The anticommutativity rule  $ab = (-1)^{|a||b|}ba$  gives  $a^2 = -a^2$ , so  $2a^2 = 0$ . This takes place in  $H^2(X \times Y)$ , which is torsion-free, so  $a^2 = 0$  as required.

## 6. CONFIGURATION SPACES

**Problem 6.1.** Consider the space

$$X = F_4\mathbb{C} = \{(z_0, z_1, z_2, z_3) \in \mathbb{C}^4 \mid z_i \neq z_j \text{ whenever } i \neq j\}.$$

The cohomology of  $X$  was described in lectures in terms of generators and relations. Use this to give a basis for  $H^*(X)$ . (You can check your answer against the following facts:  $H^*(F_n\mathbb{C})$  has total rank  $n!$ , whereas the group  $H^{n-1}(F_n\mathbb{C})$  has rank  $(n-1)!$ , and the groups  $H^m(F_n\mathbb{C})$  are zero for  $m \geq n$ .)

**Solution:** Recall that the generators are of the form  $a_{ij}$  where  $i \neq j$  and  $0 \leq i, j < 4$ ; the relations are  $a_{ij} = a_{ji}$  and  $a_{ij}^2 = 0$  together with relations

$$\rho_{ijk} : \quad a_{ij}a_{jk} + a_{jk}a_{ki} + a_{ki}a_{ij} = 0$$

for all distinct triples  $i, j, k$ . It is now easy to see that

$$\begin{aligned} H^0(F_4\mathbb{C}) &= \mathbb{Z} \\ H^1(F_4\mathbb{C}) &= \mathbb{Z}\{a_{01}, a_{02}, a_{03}, a_{12}, a_{13}, a_{23}\}. \end{aligned}$$

Next, if we reorder the triple  $i, j, k$  then the above relation  $\rho_{ijk}$  changes at most by an overall sign. We can therefore restrict attention to triples in which  $i < j < k$ , and in that case we rewrite the relation (using the graded-commutativity rule) as

$$a_{ij}a_{ik} = a_{ij}a_{jk} - a_{ik}a_{jk}.$$

We can use this to eliminate terms that involve the smallest index  $i$  twice in terms of monomials that involve  $i$  at most once. Using this repeatedly, we can rewrite any monomial in terms of monomials in which the lower index is never repeated. This gives

$$\begin{aligned} H^2(F_4\mathbb{C}) &= \mathbb{Z}\{a_{01}a_{12}, a_{01}a_{13}, a_{01}a_{23}, a_{02}a_{12}, a_{02}a_{13}, a_{02}a_{23}, a_{03}a_{12}, a_{03}a_{13}, a_{03}a_{23}, a_{12}a_{23}, a_{13}a_{23}\} \\ H^3(F_4\mathbb{C}) &= \mathbb{Z}\{a_{01}a_{12}a_{23}, a_{01}a_{13}a_{23}, a_{02}a_{12}a_{23}, a_{02}a_{13}a_{23}, a_{03}a_{12}a_{23}, a_{03}a_{13}a_{23}\}. \end{aligned}$$

**Problem 6.2.** In the notes we defined spaces  $F_n\mathbb{C}$  and elements  $a_{ij} \in H^1(F_n\mathbb{C})$ . Prove that  $a_{ij}a_{jk}a_{ki} = 0$  in  $H^3(F_n\mathbb{C})$ . Then prove that  $a_{ij}a_{jk}a_{km}a_{mi} = 0$  in  $H^4(F_n\mathbb{C})$ . Then give an inductive proof of the obvious generalisation.

**Solution:**

(a) Recall that

$$a_{ij}a_{jk} + a_{jk}a_{ki} + a_{ki}a_{ij} = 0.$$

Multiply this equation on the right by  $a_{ki}$  and recall that  $a_{ki}^2 = 0$  to get  $a_{ij}a_{jk}a_{ki} = 0$ .

(b) Next, multiply the displayed equation instead by  $a_{km}a_{mi}$  to get

$$a_{ij}a_{jk}a_{km}a_{mi} + a_{jk}a_{ki}a_{km}a_{mi} + a_{ki}a_{ij}a_{km}a_{mi} = 0.$$

Here the second term and third terms are divisible by  $a_{ki}a_{mi}a_{km}$ , which is the same as  $a_{ki}a_{im}a_{mk}$ , which vanishes by (a). It follows that the first term must vanish as well, so  $a_{ij}a_{jk}a_{km}a_{mi} = 0$ .

(c) More generally, for any list  $u = (i, j, k, \dots, l, m, n)$  of indices (with length at least four), put

$$c(u) = a_{ij}a_{jk} \cdots a_{lm}a_{mn}.$$

We claim that if the first and last entries  $i$  and  $n$  are the same, then  $c(u) = 0$ . We have checked this in the case where  $u$  has length 4 or 5. If  $u$  is longer than that, we multiply the relation  $a_{ij}a_{jk} + a_{jk}a_{ki} + a_{ki}a_{ij} = 0$  by the element  $b = c(k, \dots, l, m, n)$  (where  $n = i$  by assumption). After remembering that  $a_{ki} = a_{ik}$  and that elements of degree one anticommute, we get

$$a_{ij}a_{jk}b = (a_{ij} - a_{jk})a_{ik}b.$$

The left hand side is just  $c(u)$ , and the right hand side is  $(a_{ij} - a_{jk})c(v)$ , where  $v = (i, k, \dots, l, m, n)$ . Here  $v$  is shorter than  $u$  and the first and last entries are the same so we can assume inductively that  $c(v) = 0$ . The displayed equation then gives  $c(u) = 0$  as required.

**Problem 6.3.** Recall that  $B_n\mathbb{C}$  is the set of subsets  $S \subset \mathbb{C}$  such that  $|S| = n$  (topologised as a quotient of  $F_n\mathbb{C}$ ). Prove that  $B_2\mathbb{C}$  is homotopy equivalent to  $S^1$ .

**Solution:** Use the maps  $f: S^1 \rightarrow B_2\mathbb{C}$  and  $g: B_2\mathbb{C} \rightarrow S^1$  given by  $f(z) = \{w \in \mathbb{C} \mid w^2 = z\}$  and  $g(\{a, b\}) = (a - b)^2 / |a - b|^2$ .

**Problem 6.4.** Construct homeomorphisms

$$F_2\mathbb{C} \simeq \mathbb{C} \times \mathbb{C}^\times$$

$$F_3\mathbb{C} \simeq \mathbb{C} \times \mathbb{C}^\times \times (\mathbb{C} \setminus \{0, 1\})$$

$$B_2\mathbb{C} \simeq \mathbb{C} \times \mathbb{C}^\times$$

Describe the cohomology of all these spaces.

**Solution:** The map  $\phi(x, y) = (x, y - x)$  gives a homeomorphism  $F_2\mathbb{C} \rightarrow \mathbb{C} \times \mathbb{C}^\times$ , with inverse  $\phi^{-1}(u, v) = (u, u + v)$ . The map  $\psi(x, y, z) = (x, y - x, (z - x)/(y - x))$  gives a homeomorphism  $F_3\mathbb{C} \rightarrow \mathbb{C} \times \mathbb{C}^\times \times (\mathbb{C} \setminus \{0, 1\})$  with inverse  $\psi^{-1}(u, v, w) = (u, u + v, u + vw)$ . The map  $\chi(\{x, y\}) = (x + y, (x - y)^2)$  gives a homeomorphism  $B_2\mathbb{C} \rightarrow \mathbb{C} \times \mathbb{C}^\times$ , with inverse

$$\chi^{-1}(u, v) = \{z \in \mathbb{C} \mid (2z - u)^2 = v\}.$$

If  $V \subset \mathbb{C}$  is simply connected then we can choose a holomorphic branch of the square root function on  $V$  and for  $(u, v) \in \mathbb{C} \times V$  we have  $\chi^{-1}(u, v) = \{(u + \sqrt{v})/2, (u - \sqrt{v})/2\}$ ; this proves that  $\chi^{-1}$  is continuous. We have  $H^*\mathbb{C} = \mathbb{Z}$  and  $H^*\mathbb{C}^\times = \mathbb{Z}\{1, a\}$  and  $H^*(\mathbb{C} \setminus \{0, 1\}) = \mathbb{Z}\{1, b, c\}$ , where  $|a| = |b| = |c| = 1$  and  $a^2 = b^2 = c^2 = bc = 0$ . It follows that  $H^*F_2\mathbb{C} \simeq \mathbb{Z}[a]/a^2 = \mathbb{Z}\{1, a\} \simeq H^*B_2\mathbb{C}$ , and that  $H^*F_3\mathbb{C} = \mathbb{Z}[a, b, c]/(a^2, b^2, c^2, bc) = \mathbb{Z}\{1, a, b, c, ab, ac\}$ .

**Problem 6.5.** Let  $F_2\mathbb{R}^n$  denote the space of pairs  $(a, b)$  with  $a, b \in \mathbb{R}^n$  and  $a \neq b$ . Let  $B_2\mathbb{R}^n$  be the quotient of  $F_2\mathbb{R}^n$  by the evident action of  $C_2$ , so  $(a, b) \sim (c, d)$  iff  $((a, b) = (c, d)$  or  $(a, b) = (d, c)$ ). Let  $\mathbb{R}P^{n-1}$  denote the space of one-dimensional subspaces  $L \leq \mathbb{R}^n$ . Show that  $B_2\mathbb{R}^n$  is homotopy equivalent to  $\mathbb{R}P^{n-1}$ .

**Solution:** Define  $\phi: F_2\mathbb{R}^n \rightarrow \mathbb{R}^n \times S^{n-1} \times \mathbb{R}^+$  by  $\phi(x, y) = (x + y, (x - y)/\|x - y\|, \|x - y\|)$ . This is a homeomorphism, with inverse  $\phi^{-1}(u, v, r) = ((u + rv)/2, (u - rv)/2)$ . Moreover, if  $\phi(x, y) = (u, v, r)$  then  $\phi(y, x) = (u, -v, r)$ . Note that  $B_2\mathbb{R}^n$  is the quotient of  $F_2\mathbb{R}^n$  by the relation  $(x, y) \sim (y, x)$ , and  $\mathbb{R}P^{n-1}$  is the quotient of  $S^{n-1}$  by the relation  $v \sim -v$ . It follows that  $\mathbb{R}^n \times \mathbb{R}P^{n-1} \times \mathbb{R}^+$  is the quotient of  $\mathbb{R}^n \times S^{n-1} \times \mathbb{R}^+$  by the relation  $(u, v, r) \sim (u, -v, r)$ ; this uses a small lemma in point-set topology and relies on the fact that  $\mathbb{R}^n \times \mathbb{R}^+$  is locally compact Hausdorff. We thus get an induced homeomorphism  $B_2\mathbb{R}^n \simeq \mathbb{R}^n \times \mathbb{R}P^{n-1} \times \mathbb{R}^+$ , and this is visibly homotopy equivalent to  $\mathbb{R}P^{n-1}$ .

## 7. MATRIX GROUPS

**Problem 7.1.** If  $V$  and  $W$  are finite-dimensional complex vector spaces, we let  $\mathcal{J}(V, W)$  be the space of injective linear maps  $V \rightarrow W$ . Describe the spaces  $\mathcal{J}(\mathbb{C}, \mathbb{C})$ ,  $\mathcal{J}(\mathbb{C}, \mathbb{C}^2)$ ,  $\mathcal{J}(\mathbb{C}^2, \mathbb{C})$  and  $\mathcal{J}(\mathbb{C}^2, \mathbb{C}^2)$  up to homotopy, and describe their cohomology, including the ring structure.

**Solution:**

- (a) An injective linear map  $f: \mathbb{C} \rightarrow \mathbb{C}$  has the form  $f(z) = zu$  for some nonzero  $u \in \mathbb{C}$ , so  $\mathcal{J}(\mathbb{C}, \mathbb{C}) \simeq \mathbb{C} \setminus \{0\} \cong S^1$ , so  $H^*(\mathcal{J}(\mathbb{C}, \mathbb{C})) \simeq E[u_1]$  with  $|u_1| = 1$ .
- (b) An injective linear map  $f: \mathbb{C} \rightarrow \mathbb{C}^2$  has the form  $f(z) = zu$  for some nonzero  $u \in \mathbb{C}^2$ , so  $\mathcal{J}(\mathbb{C}, \mathbb{C}^2) \simeq \mathbb{C}^2 \setminus \{0\} \cong S^3$ , so  $H^*(\mathcal{J}(\mathbb{C}, \mathbb{C}^2)) \simeq E[u_3]$  with  $|u_3| = 3$ .
- (c) There are no injective linear maps from  $\mathbb{C}^2$  to  $\mathbb{C}$ , so  $\mathcal{J}(\mathbb{C}^2, \mathbb{C}) = \emptyset$ , and  $H^*(\mathcal{J}(\mathbb{C}^2, \mathbb{C})) = 0$ .
  - (d)] We have  $\mathcal{J}(\mathbb{C}^2, \mathbb{C}^2) = GL_2(\mathbb{C})$ , and we have seen that this is homotopy equivalent to  $U(2)$ , so  $H^*(\mathcal{J}(\mathbb{C}^2, \mathbb{C}^2)) \simeq E[a_1, a_3]$ .

**Problem 7.2.** Give a path joining  $I$  to  $-I$  in  $U(2)$ .

**Solution:** The two obvious answers are  $u(t) = \begin{bmatrix} e^{i\pi t} & 0 \\ 0 & e^{i\pi t} \end{bmatrix}$  or  $v(t) = \begin{bmatrix} \cos(\pi t) & -\sin(\pi t) \\ \sin(\pi t) & \cos(\pi t) \end{bmatrix}$ .

**Problem 7.3.** Put  $SU(n) = \{A \in U(n) \mid \det(A) = 1\}$ . Define  $\alpha: SU(3) \rightarrow S^5 \times S^5$  by  $\alpha(A) = (Ae_0, Ae_1)$  (where  $\{e_0, e_1, e_2\}$  is the standard basis of  $\mathbb{C}^3$ ). Prove that  $\alpha$  is injective but not surjective.

**Solution:** If  $A \in U(n)$  then  $\langle Au, Av \rangle = \langle u, v \rangle$  for all  $u$  and  $v$ . In particular, we have  $\langle Ae_0, Ae_1 \rangle = 0$ , so

$$\text{image}(\alpha) \subseteq \{(a, b) \in S^5 \times S^5 \mid \langle a, b \rangle = 0\}.$$

In particular,  $(e_0, e_0) \notin \text{image}(\alpha)$ , so  $\alpha$  is not surjective.

Now suppose we have  $A, A' \in SU(3)$  with  $\alpha(A) = \alpha(A')$ , so  $Ae_0 = A'e_0$  and  $Ae_1 = A'e_1$ . Put  $B = A^{-1}A'$ , and note that  $Be_0 = e_0$  and  $Be_1 = e_1$ . We know that  $Be_2$  must be orthogonal to  $Be_0 = e_0$  and  $Be_1 = e_1$ , so  $Be_2 = \lambda e_2$  for some  $\lambda$ . We also have  $\det(B) = \det(A)^{-1} \det(A') = 1$ , and it follows that  $\lambda = 1$ . This means that  $B = I$  and so  $A = A'$ , proving that  $\alpha$  is injective.

**Problem 7.4.** Prove that  $SU(2)$  is homeomorphic to  $S^3$ , and thus that  $U(2)$  is homeomorphic to  $S^1 \times S^3$ .

**Solution:** Recall that

$$SU(2) = \{A \in M_2(\mathbb{C}) \mid A^\dagger A = I \text{ and } \det(A) = 1\}.$$

More explicitly, if  $A = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$  then

$$A^\dagger A = \begin{bmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \end{bmatrix} \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} |a|^2 + |b|^2 & \bar{a}c + \bar{b}d \\ a\bar{c} + b\bar{d} & |c|^2 + |d|^2 \end{bmatrix}.$$

The conditions are  $|a|^2 + |b|^2 = |c|^2 + |d|^2 = 1$  and  $a\bar{c} + b\bar{d} = 0$  and  $ad - bc = 1$ .

We will identify  $S^1$  with  $\{a \in \mathbb{C} \mid |a| = 1\}$ , and  $S^3$  with  $\{(a, b) \in \mathbb{C}^2 \mid |a|^2 + |b|^2 = 1\}$ . We can thus define  $f: SU(2) \rightarrow S^3$  by  $f\left[\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right] = (a, b)$ , and we can define  $g: S^3 \rightarrow SU(2)$  by  $g(a, b) = \begin{bmatrix} a & -\bar{b} \\ b & \bar{a} \end{bmatrix}$ . Clearly  $fg = 1_{S^3}$ . Suppose  $A \in SU(2)$  and put  $B = A^{-1}.gf(A)$ . By construction,  $gf(A)$  has the same first column as  $A$ , and it follows that the first column of  $B$  is  $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ , and also  $B \in SU(2)$ . From the above equations describing  $SU(2)$ , we see that  $B = I$  and so  $gf(A) = A$ . Thus  $f$  and  $g$  are mutually inverse homeomorphisms.

Now define  $j: S^1 \rightarrow U(2)$  by  $j(a) = \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}$ , so  $\det(j(a)) = a$ . Note that for  $B \in U(2)$  we have

$$\det(j(\det(B))^{-1}.B) = \det(B)^{-1} \cdot \det(B) = 1,$$

so  $j(\det(B))^{-1}.B \in SU(2)$ . Define  $p: S^1 \times SU(2) \rightarrow U(2)$  by  $p(a, A) = j(a)A$ , and define  $q: U(2) \rightarrow S^1 \times SU(2)$  by  $q(B) = (\det(B), j(\det(B))^{-1}.B)$ . One can then check directly that  $pq = 1_{U(2)}$  and  $qp = 1_{S^1 \times SU(2)}$ , so  $U(2) \simeq S^1 \times SU(2) \simeq S^1 \times S^3$ .

**Problem 7.5.** Prove that the space  $GL_2^+(\mathbb{R}) = \{A \in GL_2(\mathbb{R}) \mid \det(A) > 0\}$  is homeomorphic to  $\mathbb{R}^3 \times S^1$ .

**Solution:** A direct, but not very illuminating proof is as follows. Define  $f: \mathbb{R}^3 \times S^1 \rightarrow GL_2^+(\mathbb{R})$  by

$$f(u, v, w, (a, b)) = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} e^u & v \\ 0 & e^w \end{bmatrix} = \begin{bmatrix} ae^u & av - be^w \\ be^u & bv + ae^w \end{bmatrix}.$$

Define  $g: GL_2^+(\mathbb{R}) \rightarrow \mathbb{R}^3 \times S^1$  as follows. Given a matrix  $A = \begin{bmatrix} p & q \\ r & s \end{bmatrix} \in GL_2^+(\mathbb{R})$ , note that  $(p, r)$  cannot be  $(0, 0)$ , so  $p^2 + r^2 > 0$ , and also  $ps - qr = \det(A) > 0$  by assumption. We put

$$g(A) = \left[ \ln(\sqrt{p^2 + r^2}), \frac{pq + rs}{\sqrt{p^2 + r^2}}, \ln\left(\frac{ps - rq}{\sqrt{p^2 + r^2}}\right), \frac{(p, q)}{\sqrt{p^2 + r^2}} \right].$$

One checks directly that  $f$  and  $g$  are inverse to each other.

More conceptually, one can combine the homeomorphism  $SO(2) = S^1$  with the factorisation implicit in the Gram-Schmidt procedure.

**Problem 7.6.** Put  $J = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$ , and let  $G$  be the group of matrices  $A \in GL_4(\mathbb{R})$  such that  $A^T J A = J$ . This is called the *Lorenz group*. Prove that it has at least four path-components.

**Solution:** First, the matrices  $I$ ,  $-I$ ,  $J$  and  $-J$  lie in  $G$ . We claim that they lie in different path components.

As a first step towards this, note that for  $A \in G$  we have

$$-1 = \det(J) = \det(A^T J A) = \det(A^T) \det(J) \det(A) = -\det(A)^2,$$

so  $\det(A) = \pm 1$ . As  $\det: G \rightarrow \{\pm 1\}$  is continuous it must be constant on each path component. As  $\det(\pm I) = 1$  and  $\det(\pm J) = -1$  we see that the path components of  $I$  and  $-I$  are different from those of  $J$  and  $-J$ .

To go further, we need a more detailed description of  $G$ . Any matrix  $A \in M_4(\mathbb{R})$  can be written in block form as  $A = \begin{bmatrix} B & u \\ v^T & c \end{bmatrix}$ , where  $B \in M_3(\mathbb{R})$  and  $u, v \in \mathbb{R}^3$  and  $c \in \mathbb{R}$ . We then find that

$$A^T J A = \begin{bmatrix} B^T B - vv^T & B^T u - cv \\ u^T B - cv^T & -c^2 + u^T u \end{bmatrix}.$$

Thus, for  $A \in G$  we must have  $-c^2 + u^T u = -1$ , so  $c^2 = 1 + u^T u = 1 + \|u\|^2$ . This is clearly impossible if  $c = 0$ . Thus, we see that for all  $A \in G$ , the bottom right entry  $A_{44}$  is nonzero. We therefore have a continuous map  $f: G \rightarrow \{\pm 1\}$  given by

$$f(A) = \frac{A_{44}}{|A_{44}|} = \begin{cases} 1 & \text{if } A_{44} > 0 \\ -1 & \text{if } A_{44} < 0. \end{cases}$$

(This is actually a group homomorphism, but we will not need that.) We can then define  $g: G \rightarrow \{\pm 1\}^2$  by  $g(A) = (\det(A), f(A))$ . This has  $g(I) = (1, 1)$  and  $g(-I) = (1, -1)$  and  $g(J) = (-1, -1)$  and  $g(-J) = (-1, 1)$ , so the path components of  $I, -I, J$  and  $-J$  are all different.

**Problem 7.7.** Recall the complex reflection map  $\rho: S^1 \times \mathbb{C}P^1 \rightarrow U(2)$ : the matrix  $\rho(z, L)$  has eigenvalue  $z$  on  $L$ , and eigenvalue 1 on  $L^\perp$ . Consider the following two matrices:

$$A = \frac{1}{2} \begin{bmatrix} i+1 & -i-1 \\ i+1 & i+1 \end{bmatrix} \quad B = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$$

One of these has the form  $\rho(z, L)$  for some  $z$  and  $L$ , and the other does not lie in the image of  $\rho$ . Work out which is which, and find  $z$  and  $L$ .

**Solution:** The characteristic polynomial of  $\rho(z, L)$  is  $(t-1)(t-z)$ , so the trace is  $1+z$  and the determinant is  $z$ . On the other hand, we have

$$\begin{aligned} \det(A) &= \frac{1}{4} ((1+i)^2 + (1+i)^2) = i & \det(B) &= \frac{1}{2} (1+1) = 1 \\ \text{trace}(A) &= \frac{1}{2} (1+i+1+i) = 1+i & \text{trace}(B) &= \frac{1}{\sqrt{2}} (1+1) = \sqrt{2}. \end{aligned}$$

This makes it clear that  $B$  does not have the form  $\rho(z, L)$ , but that  $A$  could be equal to  $\rho(i, L)$  for some  $L \in \mathbb{C}P^1$ . Specifically,  $L$  should be the kernel of the matrix

$$A' = A - iI = \frac{1}{2} \begin{bmatrix} 1-i & -1-i \\ 1+i & 1-i \end{bmatrix} = \frac{1-i}{2} \begin{bmatrix} 1 & -i \\ i & 1 \end{bmatrix}.$$

It follows that  $L$  must be  $\mathbb{C}u_1$ , where  $u_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} i \\ 1 \end{bmatrix}$ . The orthogonal complement is  $\mathbb{C}u_2$ , where  $u_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} -i \\ 1 \end{bmatrix}$ . One can check directly that  $Au_2 = u_2$ , and it follows that  $A$  is indeed equal to  $\rho(i, L)$ .

**Problem 7.8.** Give a formula for the rank of the free abelian group  $H^*U(n)$ .

**Solution:** We know that  $H^*U(n)$  is the exterior algebra generated by  $n$  generators  $a_1, a_3, \dots, a_{2n-1}$ . For any subset  $S$  of these generators, let  $a_S$  be the product of the elements in the subset. These elements form a basis for  $H^*U(n)$  over  $\mathbb{Z}$ , and there are  $2^n$  of them, so  $\text{rank}(H^*U(n)) = 2^n$ .

**Problem 7.9.** Give a basis for  $\tilde{H}^*(U(4)/U(2))$ . (This should be interpreted as the space obtained from  $U(4)$  by collapsing  $U(2)$  to a point, not the coset space.)

**Solution:**  $\{a_1, a_1a_3, a_1a_5, a_1a_7, a_1a_3a_5, a_1a_3a_7, a_1a_5a_7, a_1a_3a_5a_7, a_3, a_3a_5, a_3a_7, a_3a_5a_7\}$ .

**Problem 7.10.** Find an integer  $n$  and a class  $u \in H^*U(n)$  such that  $u^2$  is a nonzero element of  $H^{n^2}U(n)$ .

**Solution:** Take  $n = 4$  and  $u = a_1a_7 + a_3a_5 \in H^8U(4)$ ; then  $u^2 = 2a_1a_3a_5a_7$ , which is nonzero in  $H^{16}U(4) = \mathbb{Z}a_1a_3a_5a_7$ .

**Problem 7.11.** Let  $\mu: U(3) \times U(3) \rightarrow U(3)$  be given by  $\mu(A, B) = AB$ . By quoting facts about  $\mu^*$  proved in lectures, calculate  $\mu^*(a_1a_3a_5) \in H^9(U(3) \times U(3))$ .

**Solution:** Write  $b_i = a_i \otimes 1$  and  $c_i = 1 \otimes a_i$  so  $H^*(U(3) \times U(3))$  is the exterior algebra generated by  $b_1, b_3, b_5, c_1, c_3, c_5$ . The generators  $a_i$  are primitive, which means that  $\mu^*(a_i) = b_i + c_i$ . The map  $\mu^*$  is a ring homomorphism, so

$$\begin{aligned} \mu^*(a_1a_3a_5) &= (b_1 + c_1)(b_3 + c_3)(b_5 + c_5) \\ &= b_1b_3b_5 + b_1b_3c_5 + b_1c_3b_5 + b_1c_3c_5 + c_1b_3b_5 + c_1b_3c_5 + c_1c_3b_5 + c_1c_3c_5 \\ &= b_1b_3b_5 + b_1b_3c_5 - b_1b_5c_3 + b_1c_3c_5 + b_3b_5c_1 - b_3c_1c_5 + b_5c_1c_3 + c_1c_3c_5. \end{aligned}$$

**Problem 7.12.** Exhibit a homeomorphism  $U(n) = S^1 \times SU(n)$ , and check that it is not an isomorphism of groups unless  $n = 1$ . Prove that

$$H^*SU(n) = (H^*U(n))/a_1 = E[a_3, a_5, \dots, a_{2n-1}].$$

**Solution:** Given a unit complex number  $z \in S^1$ , define  $j(z): \mathbb{C}^n \rightarrow \mathbb{C}^n$  by  $j(z)e_1 = ze_1$  and  $j(z)e_k = e_k$  for  $k > 1$ . Then  $j(z) \in U(n)$  and  $\det(j(z)) = z$ . Define  $\phi: S^1 \times SU(n) \rightarrow U(n)$  by  $\phi(z, \alpha) = j(z)\alpha$ . This is a homeomorphism, with inverse  $\phi^{-1}(\beta) = (\det(\beta), j(\det(\beta)^{-1})\beta)$ . However, we claim that  $U(n)$  is not isomorphic to  $S^1 \times SU(n)$  as groups (provided  $n > 1$ ). To see this, let  $A$  be the group of central elements of order  $n$  in  $U(n)$ , and let  $B$  be the analogous subgroup of  $S^1 \times SU(n)$ ; it will suffice to show that  $|A| = n$  and  $|B| = n^2$ . If we put  $\omega = e^{2\pi i/n}$  and write  $\iota$  for the  $n \times n$  identity matrix, then we have elements  $\omega^a \iota$  in  $A$  and  $(\omega^b, \omega^c \iota)$  in  $B$  (where  $0 \leq a, b, c < n$ ); we need only check that there are no more elements of  $A$  or  $B$ . Suppose  $\alpha \in A$ , and choose a normalised eigenvector  $u$  with  $\|u\| = 1$  and  $\alpha(u) = \lambda u$  for some  $\lambda \in \mathbb{C}$ . For any other  $v \in V$  with  $\|v\| = 1$ , we can choose  $\beta \in U(n)$  with  $\beta(u) = v$ . As  $\alpha$  is central we have  $\alpha(v) = \alpha\beta(u) = \beta\alpha(u) = \beta(\lambda u) = \lambda v$ . It follows that  $\alpha = \lambda \iota$ . As  $\alpha^n = 1$  we have  $\lambda^n = 1$  so  $\lambda = \omega^a$  for some  $a$ . A similar argument works for  $B$ .

We now turn to cohomology. Let  $X$  be any space, and let  $Y$  be a connected space such that  $H^k(Y)$  is a finitely generated free Abelian group for all  $k$ . Choose  $y_0 \in Y$  and define  $i: X \rightarrow X \times Y$  by  $i(x) = (x, y_0)$ ; the homotopy class of  $i$  is clearly well-defined. Let  $X \xrightarrow{p} X \times Y \xrightarrow{q} Y$  be the projections, and let  $\epsilon: H^*Y \rightarrow H^*\{y_0\} = \mathbb{Z}$  be the restriction map. Clearly  $i^*(p^*(a)q^*(b)) = (pi)^*(a)(qi)^*(b) = a\epsilon(b)$ . It follows easily from this and the Künneth Theorem that  $i^*: H^*(X \times Y) \rightarrow H^*X$  induces an isomorphism  $H^*(X \times Y)/I \rightarrow H^*X$ , where  $I$  is the ideal generated by  $q^*\tilde{H}^*Y$ . Using our homeomorphism  $U(n) \simeq S^1 \times SU(n)$ , we deduce that the inclusion  $SU(n) \rightarrow U(n)$  induces an isomorphism  $H^*U(n)/\det^*\tilde{H}^*S^1 \rightarrow H^*SU(n)$ . We know that  $H^*U(n) = E[a_1, \dots, a_{2n-1}]$ , and the generator  $a_1$  is the unique element whose restriction to  $U(1)$  is the pullback along the evaluation map  $\epsilon: U(1) \rightarrow S^1$  of the generator  $u_1 \in \tilde{H}^1S^1$ . It is easy to identify  $\epsilon$  with  $\det|_{U(1)}$  and it follows that  $H^*SU(n) = H^*U(n)/(a_1) = E[a_3, \dots, a_{2n-1}]$ .

**Problem 7.13.** It is known that any manifold  $M$  can be embedded as a subspace of a finite dimensional vector space. As an example, exhibit an embedding of  $PV$  in  $\text{Hom}(V, V)$  (for any Hermitian space  $V$ ). (Look through the discussion of the topology of  $U(V)$  for hints.)

**Solution:** Given a point  $L \in PV$ , let  $i(L): V \rightarrow V$  be the map

$$V = L \oplus L^\perp \xrightarrow{1 \oplus 0} L \oplus L^\perp = V.$$

Note that  $L$  is the image of  $i(L)$ , so  $i$  gives an injective function  $PV \rightarrow \text{Hom}(V, V)$ . To see that this is continuous, let  $q: \tilde{V} \rightarrow PV$  be the quotient map  $v \mapsto \mathbb{C}v$ , and put  $j = iq$ . We then see that  $j(v)(w) = \langle w, v \rangle / v^2$ , which easily implies that  $j: \tilde{V} \rightarrow \text{Hom}(V, V)$  is continuous. As  $iq$  is continuous and  $q$  is a quotient map we see that  $i$  is continuous.

Alternatively, one could let  $j(L)$  be the map  $(-1) \oplus 1$  with respect to the splitting  $V = L \oplus L^\perp$ . Then  $j$  can be related to the complex reflection map  $\rho: S^1 \times PV$  that we used to analyse  $U(V)$ , by the formula  $j(L) = \rho(-1, L)$ . Moreover, we have  $j(L) = id_V - 2i(L)$ , so  $i$  and  $j$  are also closely related.

**Problem 7.14.** For a complex matrix  $A$  we write  $A^\dagger$  for the conjugate transpose, so  $A_{ij}^\dagger = \overline{A_{ji}}$ . Recall that  $SU(2)$  is the group of  $2 \times 2$  complex matrices  $A$  such that  $A^\dagger A = I$  and  $\det(A) = 1$ .

- Consider a matrix  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_2\mathbb{C}$ . In terms of the entries  $a, b, c$  and  $d$ , what are the conditions for  $A$  to be an element of  $SU(2)$ ?
- Show that if  $A \in SU(2)$  and the first column of  $A$  is  $e_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$  then  $A = I$ . Deduce that if  $A, B \in SU(2)$ , and  $A$  and  $B$  have the same first column, then  $A = B$ .
- Show that there is a homeomorphism  $f: S^3 \rightarrow SU(2)$  given by

$$f((w, x, y, z)) = \begin{bmatrix} w + ix & -y + iz \\ y + iz & w - ix \end{bmatrix}.$$

(Do not assume any properties of this formula without checking them.) Note that this means that  $S^3$  can be identified with  $SU(2)$  and so has a group structure.

- Now put

$$F_3S^3 = \{(u, v, w) \in S^3 \times S^3 \times S^3 \mid u \neq v, u \neq w, v \neq w\}$$

Prove that this space is homeomorphic to  $S^3 \times \mathbb{R}^3 \times (\mathbb{R}^3 \setminus \{0\})$  and so is homotopy equivalent to  $S^3 \times S^2$ .

(**Hint:** use the group structure from (a), and stereographic projection.)

- Deduce a description of the cohomology of  $F_3S^3$ .

**Solution:**

- (a) For a matrix  $A = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$  we have  $A^\dagger = \begin{bmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \end{bmatrix}$  so

$$A^\dagger A = \begin{bmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \end{bmatrix} \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} |a|^2 + |b|^2 & \bar{a}c + \bar{b}d \\ a\bar{c} + b\bar{d} & |c|^2 + |d|^2 \end{bmatrix}.$$

Note that the top right entry is the conjugate of the bottom left entry, so if one is zero then so is the other. We also have  $\det(A) = ad - bc$ . Thus, we have  $A \in SU(2)$  iff  $|a|^2 + |b|^2 = |c|^2 + |d|^2 = ad - bc = 1$  and  $\bar{a}c + \bar{b}d = 0$ .

- (b) Now suppose that the first column of  $A$  is  $e_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ , or equivalently  $Ae_0 = e_0$ , so  $a = 1$  and  $b = 0$ . The conditions for  $A \in SU(2)$  then become  $|c|^2 + |d|^2 = d = 1$  and  $c = 0$ , which means that  $(a, b, c, d) = (1, 0, 0, 1)$  and  $A = I$ .

Now suppose instead that  $A, B \in SU(2)$  and that  $A$  and  $B$  have the same first column, or in other words  $Ae_0 = Be_0$ . This means that the matrix  $C = B^{-1}A$  has  $Ce_0 = e_0$  so  $C = I$  by the previous paragraph so  $A = B$ .

- (c) Define  $f: S^3 \rightarrow M_2\mathbb{C}$  and  $g: SU(2) \rightarrow \mathbb{R}^4$  by

$$f((w, x, y, z)) = \begin{bmatrix} w + ix & -y + iz \\ y + iz & w - ix \end{bmatrix}$$

$$g\left(\begin{bmatrix} a & c \\ b & d \end{bmatrix}\right) = (\operatorname{Re}(a), \operatorname{Im}(a), \operatorname{Re}(b), \operatorname{Im}(b)).$$

These are clearly continuous.

For  $(w, x, y, z) \in S^3$  we have  $w^2 + x^2 + y^2 + z^2 = 1$ . If we put  $(a, b, c, d) = (w + ix, y + iz, -y + iz, w - ix)$  we get

$$\begin{aligned} |a|^2 + |b|^2 &= w^2 + x^2 + y^2 + z^2 = 1 \\ |c|^2 + |d|^2 &= y^2 + z^2 + w^2 + x^2 = 1 \\ ad - bc &= (w^2 + x^2) - (-y^2 - z^2) = 1 \\ a\bar{c} + b\bar{d} &= (w + ix)(-y - iz) + (y + iz)(w + ix) = 0. \end{aligned}$$

From this we see that  $f(w, x, y, z) \in SU(2)$ , so we can regard  $f$  as a continuous map  $S^3 \rightarrow SU(2)$ .

For  $A = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \in SU(2)$  we also have

$$\|g(A)\|^2 = \operatorname{Re}(a)^2 + \operatorname{Im}(a)^2 + \operatorname{Re}(b)^2 + \operatorname{Im}(b)^2 = |a|^2 + |b|^2 = 1.$$

Thus, we can regard  $g$  as a continuous map  $g: SU(2) \rightarrow S^3$ .

It is clear by construction that  $g(f(w, x, y, z)) = (w, x, y, z)$  and that  $f(g(A))$  has the same first column as  $A$ . Using (b) we deduce that  $f(g(A)) = A$ . Thus,  $f$  and  $g$  are mutually inverse homeomorphisms.

- (d) Recall that  $S^3$  has a group structure with identity element  $e_0$ . Let  $\rho: S^3 \setminus \{e_0\} \rightarrow \mathbb{R}^3$  be the stereographic projection homeomorphism. For  $(u, v, w) \in F_3S^3$  we note that  $u^{-1}v$  and  $u^{-1}w$  are distinct elements of  $S^3 \setminus \{e_0\}$ , so  $\rho(u^{-1}v)$  and  $\rho(u^{-1}w)$  are distinct elements of  $\mathbb{R}^3$ . We can thus define  $p: F_3S^3 \rightarrow S^3 \times \mathbb{R}^3 \times (\mathbb{R}^3 \setminus \{0\})$  by

$$p(u, v, w) = (u, \rho(u^{-1}v), \rho(u^{-1}w) - \rho(u^{-1}v)).$$

This is easily seen to be a homeomorphism, with inverse

$$p^{-1}(u, x, y) = (u, u\rho^{-1}(x), u\rho^{-1}(x + y)).$$

As  $\mathbb{R}^3$  is contractible and  $\mathbb{R}^3 \setminus \{0\}$  is homotopy equivalent to  $S^2$ , we deduce that  $F_3S^3$  is homotopy equivalent to  $S^3 \times S^2$ .

- (e) It follows using the Künneth Theorem that

$$H^*(F_3S^3) \simeq \mathbb{Z}[u_2, u_3]/(u_2^2, u_3^2) = \mathbb{Z}\{1, u_2, u_3, u_2u_3\}$$

(with  $|u_2| = 2$  and  $|u_3| = 3$ ).

**Problem 7.15.** Let  $X \subset U(3)$  be the space of  $3 \times 3$  unitary matrices whose characteristic polynomial is  $t^3 - t^2 - t + 1$ . Find a well-known space  $Y$  that is homeomorphic to  $X$  and thus describe  $H^*(X)$ . Your answer should specify the topologies used on  $X$  and  $Y$  and explain why relevant maps are continuous.

(Hint: unitary matrices are diagonalisable, and eigenvectors with distinct eigenvalues are orthogonal.)

**Solution:** We give  $X$  the usual metric topology inherited from  $U(3)$ . We take  $Y = \mathbb{C}P^2 = S^5/S^1$  and topologise this as a quotient of  $S^5$ . We define  $f_0, f_1: S^5 \rightarrow M_3(\mathbb{C})$  by  $f_0(v)_{ij} = v_i\bar{v}_j$  and  $f_1(v) = I - 2f_0(v)$ . We find that  $f_0(v)^\dagger = f_0(v)$  and  $f_1(v)^\dagger = f_1(v)$  and  $f_0(v)^2 = f_0(v)$  so  $f_1(v)^2 = I$ . This proves that  $f_1(v) \in U(3)$ . If we choose additional vectors  $v', v''$  such that  $v, v', v''$  is an orthonormal basis, we see that  $f_1(v)v = -v$  but  $f_1(v)v' = v'$  and  $f_1(v)v'' = v''$ . From this we see that the characteristic polynomial of  $f_1(v)$  is  $(t + 1)(t - 1)^2 = t^3 - t^2 - t + 1$ . We

have thus defined a continuous map  $f_1: S^5 \rightarrow X$ . Now let  $q: S^5 \rightarrow Y = \mathbb{C}P^2$  be the quotient map. If  $z$  is a complex number with  $|z| = 1$  then  $z\bar{z} = 1$  so  $f_1(zv) = f_1(v)$ . It follows that there is a unique map  $f: Y \rightarrow X$  with  $f \circ q = f_1$ . As  $Y$  has the quotient topology, we see that  $f$  is continuous.

Now consider a general element  $A \in X$ . As the characteristic polynomial is  $(t+1)(t-1)^2$ , we can choose an orthonormal basis of eigenvectors  $v, v', v''$  with  $Av = -v$  and  $Av' = v'$  and  $Av'' = v''$ . From this it follows easily that  $A = f_1(v) = f(\mathbb{C}v) = f(\ker(A+I))$ . We now see that  $f: Y \rightarrow X$  is a continuous bijection. It is also clear that  $Y$  is compact (because it is a quotient of the compact space  $S^5$ ) and  $X$  is Hausdorff (as it is a subspace of a metric space). A continuous bijection from a compact space to a Hausdorff space is automatically a homeomorphism.

We now have

$$H^*(X) \simeq H^*(\mathbb{C}P^2) = \mathbb{Z}[x]/x^3 = \mathbb{Z}\{1, x, x^2\}$$

with  $|x| = 2$ .

**Problem 7.16.** Exhibit a homeomorphism  $f: S^1 \times SU(3) \rightarrow U(3)$ . Deduce that

$$\text{rank}(H^i(U(3))) = \text{rank}(H^i(SU(3))) + \text{rank}(H^{i-1}(SU(3))),$$

and thus calculate the ranks of all the cohomology groups of  $SU(3)$ . (You may assume that these are all free abelian groups.)

**Solution:** Given a unit complex number  $z \in S^1$ , define

$$J(z) = \begin{bmatrix} z & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \in U(3),$$

so  $\det(J(z)) = z$ . Then define  $f: S^1 \times SU(3) \rightarrow U(3)$  by  $f(z, A) = J(z)A$  (so  $\det(f(z, A)) = z$ ). In the opposite direction, define  $g: U(3) \rightarrow S^1 \times SU(3)$  by  $g(B) = (\det(B), J(\det(B))^{-1}B)$ . It is then easy to check that  $g$  is inverse to  $f$ , so  $f$  is a homeomorphism.

Now put

$$\begin{aligned} \beta_i &= \text{rank}(H^i(SU(3))) \\ \gamma_i &= \text{rank}(H^i(U(3))) = \text{rank}(H^i(S^1 \times SU(3))). \end{aligned}$$

We must show that  $\gamma_i = \beta_i + \beta_{i-1}$ . Let  $b_1, \dots, b_r$  be a homogeneous basis for  $H^*(SU(3))$ . The basic version of the Künneth theorem tells us that the elements  $b_i$  and  $u_1 b_i$  give a basis for  $H^*(S^1 \times SU(3)) \simeq H^*(U(3))$ . To get basis elements for  $H^i(U(3))$  we can either take basis elements for  $H^i(SU(3))$ , or we can take basis elements for  $H^{i-1}(SU(3))$  and multiply them by  $u_1$ . This gives  $\gamma_i = \beta_i + \beta_{i-1}$  as required. We have seen that  $H^*(U(3)) \simeq E[a_1, a_3, a_5]$ , which gives the following table:

| $i$ | basis for $H^i(U(3))$ | $\gamma_i$ |
|-----|-----------------------|------------|
| 0   | 1                     | 1          |
| 1   | $a_1$                 | 1          |
| 2   |                       | 0          |
| 3   | $a_3$                 | 1          |
| 4   | $a_1 a_3$             | 1          |
| 5   | $a_5$                 | 1          |
| 6   | $a_1 a_5$             | 1          |
| 7   |                       | 0          |
| 8   | $a_3 a_5$             | 1          |
| 9   | $a_1 a_3 a_5$         | 1          |

As  $SU(3)$  is path connected we have  $\beta_0 = 1$ . We can then use the relation  $\beta_i = \gamma_i - \beta_{i-1}$  recursively to get

$$\begin{array}{ccccc} \beta_0 = 1 & \beta_1 = 0 & \beta_2 = 0 & \beta_3 = 1 & \beta_4 = 0 \\ \beta_5 = 1 & \beta_6 = 0 & \beta_7 = 0 & \beta_8 = 1 & \beta_9 = 0 \end{array}$$

(and  $\beta_i = 0$  for  $i > 9$ ).

## 8. VECTOR BUNDLES

**Problem 8.1.** If  $T$  is the tautological line bundle over  $\mathbb{C}P^n$ , prove that  $S(T \otimes T) = \mathbb{R}P^{2n+1}$ .

**Solution:** Recall that

$$S(T \otimes T) = \{(L, v) \mid L \in \mathbb{C}P^n \text{ and } v \in L \otimes L \text{ and } \|v\| = 1\}.$$

We can define  $f_0: S^{2n+1} = S(\mathbb{C}^{n+1}) \rightarrow S(T \otimes T)$  by

$$f_0(u) = (\mathbb{C}u, u \otimes u).$$

This clearly satisfies  $f_0(-u) = f_0(u)$ , so it induces a map  $f: \mathbb{R}P^{2n+1} \rightarrow S(T \otimes T)$  with  $f([u]) = f_0(u)$ . This is continuous, by the universal property of the quotient topology.

Now suppose we start with a point  $(L, v) \in S(T \otimes T)$ . Let  $u_0$  be any unit vector in the space  $L \subset \mathbb{C}^{n+1}$ . Then  $v$  and  $u_0 \otimes u_0$  are both unit vectors in the one-dimensional space  $L \otimes L$ , so  $v = e^{it}(u_0 \otimes u_0)$  for some  $t \in \mathbb{R}$ . If we put  $u = e^{it/2}u_0$  we then find that  $L = \mathbb{C}u_0 = \mathbb{C}u$  and  $v = u \otimes u$  so  $(L, v) = f_0(u)$ . Now suppose we also have  $(L, v) = f_0(w)$  for some other  $w \in S^{2n+1}$ . As  $\mathbb{C}w = L = \mathbb{C}u$  we have  $w = \lambda u$  for some  $\lambda \in \mathbb{C}$ . As  $w \otimes w = v = u \otimes u$  we have  $\lambda^2 = 1$  so  $\lambda = \pm 1$  so  $w = \pm u$ . From this it is clear that the map  $f: \mathbb{R}P^{2n+1} \rightarrow S(T \otimes T)$  is bijective. It is also easy to see that  $\mathbb{R}P^{2n+1}$  is compact and  $S(T \otimes T)$  is Hausdorff, so  $f$  is a homeomorphism.

**Problem 8.2.** Let  $q: V \setminus \{0\} \rightarrow PV$  be the usual quotient map, and let  $T$  denote the tautological bundle over  $PV$ . Prove that  $q^*T$  is isomorphic to a constant bundle.

**Solution:** For  $v \in V \setminus \{0\}$  we have  $T_{q(v)} = q(v) = \mathbb{C}v$ . It follows that

$$E(q^*T) = \{(v, w) \mid v \in V \setminus \{0\}, w \in \mathbb{C}v\}.$$

We can therefore define a homeomorphism

$$f: \mathbb{C} \times (V \setminus \{0\}) \rightarrow E(q^*T)$$

by  $f(z, v) = (v, zv)$ . This is linear on each fibre and so gives an isomorphism  $\mathbb{C} \rightarrow q^*(T)$  of vector bundles.

**Problem 8.3.** Let  $L$  be the Möbius bundle over  $S^1$ , given by  $L_z = \{w \in \mathbb{C} \mid w^2 \in z \cdot [0, \infty)\}$ . Prove that the bundle  $\mathbb{C} \otimes_{\mathbb{R}} L \simeq L \oplus L$  is isomorphic to a constant bundle.

**Solution:** We can define

$$g: E(L \oplus L) = \{(z, u, v) \in \mathbb{C}^3 \mid u^2, v^2 \in z \cdot [0, \infty)\} \rightarrow S^1 \times \mathbb{C}$$

by  $g(z, u, v) = (z, u + iv)$ . For each  $z \in S^1$  this restricts to give a linear map  $g_z: L_z \oplus L_z \rightarrow \mathbb{C}$ . If  $z = e^{i\theta}$  then  $L_z = \mathbb{C} \cdot e^{i\theta/2}$  and

$$g_z(xe^{i\theta/2}, ye^{i\theta/2}) = (x + iy)e^{i\theta}.$$

It is clear from this that  $g_z$  is an isomorphism of vector spaces for all  $z$ , and it follows that  $g$  is an isomorphism of vector bundles.

**Problem 8.4.** Let  $V$  and  $W$  be complex vector bundles over a base  $X$ , and suppose that  $\dim_{\mathbb{C}}(W) = 1$ . Prove that  $P(V \otimes W) \simeq PV$ .

**Solution:** If  $x \in X$  and  $L \in PV_x$  then  $L$  and  $W_x$  both have dimension one, so  $L \otimes W_x$  also has dimension one, and  $L \otimes W_x \leq V_x \otimes W_x = (V \otimes W)_x$ . We can therefore define a map  $f: PV \rightarrow P(V \otimes W)$  by  $f(x, L) = (x, L \otimes W_x)$ . We can also choose a nonzero vector  $w_x \in W_x$  and define an isomorphism  $g: V_x \otimes W_x \rightarrow V_x$  by  $g(v \otimes \lambda w_x) = \lambda v$ ; this gives a map  $Pg: P(V_x \otimes W_x) \rightarrow PV_x$ , which is inverse to  $f$ . It follows that  $f$  is a continuous bijection, but not yet that  $f$  is a homeomorphism; we must still prove that  $g$  is continuous. If  $W$  is trivialisable then we can choose  $w_x$  to be a continuous function of  $x$ , and the continuity of  $g$  follows easily. In general, we can cover  $X$  by open sets  $U_\alpha$  such that  $W|_{U_\alpha}$  is trivialisable, so  $g$  is continuous on each of the sets  $\pi^{-1}U_\alpha \subseteq P(V \otimes W)$ . These sets form an open cover, so  $g$  is continuous as required.

**Problem 8.5.** Let  $V$  be a Hermitian space, and let  $L$  be the tautological bundle over  $PV$ . Interpret  $\text{Hom}(L, L^\perp)$  as a bundle over  $PV$ , and show that it is isomorphic to the tangent bundle.

**Solution:** Suppose  $L \in PV$  and  $\alpha: L \rightarrow L^\perp$ . Let  $i_L: L \rightarrow V$  be the inclusion, so that  $i_L + \alpha: L \rightarrow V$ . Define  $f_L(\alpha) = (i_L + \alpha)(L) \in PV$ . Next, put  $U_L = \{M \in PV \mid M \not\subseteq L^\perp\}$ , which is a neighbourhood of  $L$ . If  $M \in U_L$  then  $V = L^\perp \oplus M$ ; thus, if  $v \in L$  then there is a unique element  $\beta(v) \in L^\perp$  such that  $v + \beta(v) \in M$ . It is not hard to check that  $\beta$  is a homomorphism, so we can define  $g_L: U_L \rightarrow \text{Hom}(L, L^\perp)$  by  $g_L(M) = \beta$ . One can now check that  $f_L$  and  $g_L$  are mutually inverse homeomorphisms. We didn't discuss in lectures how to define a smooth structure on  $PV$ , but the best way is just to take these maps  $f_L$  as charts. If we do this, we can obviously use  $f_L$  to identify the tangent space to  $PV$  at  $L$  with the tangent space to  $\text{Hom}(L, L^\perp)$  at 0, which is in turn naturally identified with the vector space  $\text{Hom}(L, L^\perp)$  itself.

**Problem 8.6.** Let  $V$  be a complex vector bundle over  $U(n)$  that can be written as a direct sum of line bundles. What can you say about  $f_V(t)$ ?

**Solution:**

**Problem 8.7.** Find a partition of unity on  $\mathbb{R}$  subordinate to the covering by open sets  $U_n = (n-2, n+2)$  for  $n \in \mathbb{Z}$ .

**Solution:** Use the functions

$$\phi_n(x) = \max(0, 1 - |x - n|) = \begin{cases} t & \text{if } x = n - 1 + t \text{ with } 0 \leq t \leq 1 \\ 1 - t & \text{if } x = n + t \text{ with } 0 \leq t \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

**Problem 8.8.** Consider the lens space  $X = S^3/C_d$ ; what is  $|\text{Pic}(X)|$ ?

**Solution:** We showed in lectures that  $\text{Pic}(X) \simeq H^2 X$  and  $H^* X = \mathbb{Z}[x]/(x^2, dx) \oplus \mathbb{Z}u$  with  $x \in H^2$  and  $u \in H^3$ . It follows that  $H^2 X = \mathbb{Z}/d$  (generated by  $x$ ) and so  $|\text{Pic}(X)| = d$ .

**Problem 8.9.** Consider the  $n$ -torus  $T^n = S^1 \times \cdots \times S^1 = (S^1)^n$ . What is the rank of the abelian group  $\text{Pic}(T^n)$ ?

**Solution:** The full cohomology ring  $H^*(T^n)$  is the exterior algebra generated by elements  $a_1, \dots, a_n$  with  $|a_i| = 1$ . We can identify  $\text{Pic}(T^n)$  with  $H^2(T^n)$ , which has a basis consisting of the products  $a_i a_j$  with  $1 \leq i < j \leq n$ . It follows that the rank is  $\binom{n}{2} = (n^2 - n)/2$ .

**Problem 8.10.** Let  $L$  be the tautological line bundle over  $\mathbb{CP}^1$ , fix an integer  $m$ , and put  $X = P(\mathbb{C} \oplus L^m)$ . For which  $m$  does there exist  $a \in H^2(X)$  such that  $a^2$  generates  $H^4(X)$ ?

**Solution:** We have  $H^*(\mathbb{CP}^1) = \mathbb{Z}[x]/x^2$ . The Chern polynomial for  $\mathbb{C} \oplus L^m$  is  $f(t) = t(t - mx)$ , so

$$H^*(X) = H^*(\mathbb{CP}^1)[y]/f(y) = \mathbb{Z}[x, y]/(x^2, y(y - mx)) = \mathbb{Z}[x, y]/(x^2, y^2 - mxy) = \mathbb{Z}\{1, x, y, xy\}.$$

Consider an element  $a = ix + jy \in H^2(X)$ . We then have

$$a^2 = i^2 x^2 + 2ijxy + j^2 y^2 = 0 + 2ijxy + j^2 mxy = j(2i + mj)xy.$$

For this to generate  $H^4(X)$  we must have  $j(2i + mj) = \pm 1$ , so  $j = \pm 1$  and  $2i + mj = \pm 1$ . If  $m$  is even then this is clearly impossible. If  $m$  is odd, say  $m = 2p + 1$ , then we can take  $i = p$  and  $j = -1$ , so we see that the element  $a = px - y$  satisfies  $a^2 = xy$ , so  $a^2$  generates  $H^4(X)$ .

## 9. MISCELLANEOUS

**Problem 9.1.** Recall that the Mobius band can be described as

$$M = \{(z, w) \in \mathbb{C} \times \mathbb{C} \mid |z| = 1 \text{ and } w^2 = tz \text{ for some } t \geq 0\}.$$

Prove that this is homotopy equivalent to  $S^1$ .

**Solution:** Define  $i: S^1 \rightarrow M$  by  $i(z) = (z, 0)$ . Define  $r: M \rightarrow S^1$  by  $r(z, w) = z$ . Define  $h: [0, 1] \times M \rightarrow M$  by

$$h(s, (z, w)) = (z, sw).$$

We then find that  $ri = 1: S^1 \rightarrow S^1$  and  $h(1, (z, w)) = (z, w)$  and  $h(0, (z, w)) = i(r(z, w))$ . This proves that  $r$  is a homotopy inverse for  $i$ , so  $i$  is a homotopy equivalence.

**Problem 9.2.** Let  $G$  be a path-connected topological group, such that  $H^*(G)$  is a finitely generated free abelian group. Prove that every element of  $H^1(G)$  is primitive.

**Solution:** As  $G$  is path-connected, the single element 1 gives a basis for  $H^0(G)$ . Choose a basis  $e_1, \dots, e_r$  for  $H^1(G)$  over  $\mathbb{Z}$ . Define  $\lambda, \rho, \mu: G^2 \rightarrow G$  by  $\lambda(g, h) = g$  and  $\rho(g, h) = h$  and  $\mu(g, h) = gh$ . We need to show that  $\mu^*(e_i) = \lambda^*(e_i) + \rho^*(e_i)$  for all  $i$ .

For each  $i$  we have external products

$$\begin{aligned} e_i \times 1 &= \lambda^*(e_i)\rho^*(1) = \lambda^*(e_i) \in H^1(G^2) \\ 1 \times e_i &= \lambda^*(1)\rho^*(e_i) = \rho^*(e_i) \in H^1(G), \end{aligned}$$

and the Künneth Theorem tells us that the list

$$\lambda^*(e_1), \dots, \lambda^*(e_n), \rho^*(e_1), \dots, \rho^*(e_n)$$

is a basis for  $H^1(G^2)$ . Thus, there are coefficients  $a_{ij}, b_{ij} \in \mathbb{Z}$  with

$$\mu^*(e_i) = \sum_j a_{ij} \lambda^*(e_j) + \sum_j b_{ij} \rho^*(e_j).$$

Now define  $\phi: G \rightarrow G^2$  by  $\phi(g) = (g, 1)$ . We then have  $\lambda\phi = \mu\phi = 1: G \rightarrow G$  but  $\rho\phi$  is constant. This means that  $\phi^*\lambda^* = \phi^*\mu^* = 1: H^1(G) \rightarrow H^1(G)$  but  $\phi^*\rho^* = 0: H^1(G) \rightarrow H^1(G)$ . We can therefore apply  $\phi^*$  to our equation for  $\mu^*(e_i)$ , giving

$$e_i = \sum_{j=1}^n a_{ij} e_j.$$

As the list  $e_1, \dots, e_n$  is a basis for  $H^1(G)$ , we get  $a_{ij} = \delta_{ij}$ . A similar argument using  $\psi: g \mapsto (1, g)$  gives  $b_{ij} = \delta_{ij}$ . Substituting this into the relation  $\mu^*(e_i) = \sum_j (a_{ij}\lambda^*(e_j) + b_{ij}\rho^*(e_j))$  gives  $\mu^*(e_i) = \lambda^*(e_i) + \rho^*(e_i)$  as required.

**Problem 9.3.** Consider an integer  $n \geq 1$  and the Milnor hypersurface  $M = H_{1,n} \subset \mathbb{CP}^1 \times \mathbb{CP}^n$  with its two projections

$$\mathbb{CP}^1 \xleftarrow{p} M \rightarrow \mathbb{CP}^n.$$

- Describe the cohomology ring  $H^*(M)$  as explicitly as possible. Your description should include the ring structure and a basis for each of the individual groups  $H^i(M)$ .
- Construct a map  $s: \mathbb{CP}^1 \rightarrow M$  with  $p \circ s = \text{id}: \mathbb{CP}^1 \rightarrow \mathbb{CP}^1$ . Explain carefully why your definition is valid.
- Show that when  $n = 1$  you also have  $s \circ p = \text{id}: H_{1,1} \rightarrow H_{1,1}$ , so  $s$  is a homeomorphism.
- Describe the ring map  $s^*: H^*(M) \rightarrow H^*(\mathbb{CP}^1)$ , in terms of the elements that you listed in part (a). (You will need to consider properties of the map  $q \circ s: \mathbb{CP}^1 \rightarrow \mathbb{CP}^n$ .) Explain why your description is compatible with the relations in  $H^*(M)$  and  $H^*(\mathbb{CP}^1)$ , being careful to cover the case  $n = 1$  as well as the case  $n > 1$ .
- Now suppose that  $n > 1$ . Suppose we have  $u, v \in H^2(M)$  with  $u^2 = 0$  and  $v^n = 0$  but  $u \neq 0$  and  $v^2 \neq 0$ . Show that there are integers  $a, b \neq 0$  with  $u = ay$  and  $v = b(y - nz)$ .
- Deduce from (e) that when  $n > 1$ ,  $M$  is not homotopy equivalent to  $\mathbb{CP}^1 \times \mathbb{CP}^{n-1}$ .

**Solution:**

- We have an element  $x \in H^2(\mathbb{CP}^1)$  and we define  $y = p^*(x) \in H^2(M)$ . We also have an element  $x \in H^2(\mathbb{CP}^n)$  and we put  $z = q^*(x)$ . From the notes we have

$$H^*(H_{m,n}) = \mathbb{Z}[y, z] / (y^{m+1}, z^n - z^{n-1}y + z^{n-2}y^2 - \dots \pm y^n).$$

We can take  $m = 1$  and discard terms divisible by  $y^2$  in the second relation to get

$$H^*(M) = \mathbb{Z}[y, z] / (y^2, z^n - z^{n-1}y) = \mathbb{Z}[y, z] / (y^2, z^{n-1}(z - y)).$$

The elements  $z^i$  and  $y z^i$  with  $0 \leq i < n$  form a basis over  $\mathbb{Z}$ . For the individual groups, we can say that

- The set  $\{1\}$  is a basis for  $H^0(M) \simeq \mathbb{Z}$ .
- For  $0 < i < n$ , the set  $\{z^i, y z^{i-1}\}$  is a basis for  $H^{2i}(M) \simeq \mathbb{Z}^2$ .
- The set  $\{y z^{n-1}\}$  is a basis for  $H^{2n}(M) \simeq \mathbb{Z}$ .
- All other cohomology groups are zero (so the empty set is a basis).

- Recall that

$$\begin{aligned} M &= \{([z], [w]) \in \mathbb{CP}^1 \times \mathbb{CP}^n \mid \sum_{i=0}^1 z_i w_i = 0\} \\ &= \{([z], [w]) \in \mathbb{CP}^1 \times \mathbb{CP}^n \mid z_0 w_0 + z_1 w_1 = 0\}. \end{aligned}$$

We can define an injective linear map  $\alpha: \mathbb{C}^2 \rightarrow \mathbb{C}^{n+1}$  by

$$\alpha(z_0, z_1) = (-z_1, z_0, 0, \dots, 0).$$

If  $L$  is a one-dimensional subspace of  $\mathbb{C}^2$ , then  $\alpha(L)$  is a one-dimensional subspace of  $\mathbb{C}^{n+1}$ , or in other words a point in  $\mathbb{CP}^n$ . From the above description of  $M$  it is clear that  $(L, \alpha(L)) \in M$ , so we can define  $s: \mathbb{CP}^1 \rightarrow M$  by  $s(L) = (L, \alpha(L))$ . Because the projection  $p: M \rightarrow \mathbb{CP}^1$  is just given by  $p(L, L') = L$ , we have  $p \circ s = \text{id}: \mathbb{CP}^1 \rightarrow \mathbb{CP}^1$ .

We could also have started by writing the formula

$$s([z]) = ([z], [-z_1 : z_0 : 0 : \dots : 0]),$$

but then we would need to explain why this is consistent with the fact that  $[z] = [\lambda z]$  for  $z \in \mathbb{C}^2 \setminus \{0\}$  and  $\lambda \in \mathbb{C}^\times$ .

- Now consider the case where  $n = 1$ , so

$$M = H_{1,1} = \{([z_0 : z_1], [w_0 : w_1]) \in \mathbb{CP}^1 \times \mathbb{CP}^1 \mid z_0 w_0 + z_1 w_1 = 0\}.$$

For any point  $x = ([z], [w])$  as above we must either have  $z_0 \neq 0$  or  $z_1 \neq 0$ . If  $z_0 \neq 0$  then  $w_0 = -z_0^{-1} z_1 w_1$  so

$$(w_0, w_1) = z_0^{-1} w_1 (-z_1, z_0) = z_0^{-1} w_1 \alpha(z_0, z_1),$$

so  $[w_0 : w_1] = \alpha([z_0 : z_1])$ . If  $z_1 \neq 0$  then

$$(w_0, w_1) = -z_1^{-1}w_0(-z_1, z_0) = -z_1^{-1}w_0\alpha(z_0, z_1),$$

so again  $[w_0 : w_1] = \alpha([z_0 : z_1])$ . In all cases we therefore have  $s(p(x)) = ([z], \alpha([z])) = ([z], [w]) = x$ , as required.

(d) We now have a ring map

$$s^*: H^*(M) = \mathbb{Z}[y, z]/(y^2, z^{n-1}(z - y)) \rightarrow \mathbb{Z}[x]/x^2 = H^*(\mathbb{CP}^1).$$

We will show that this sends both  $y$  and  $z$  to  $x$ . This means that it sends the relators  $y^2$  and  $z^{n-1}(z - y)$  to  $x^2$  and  $x^{n-1}(x - x)$ , both of which are zero, so everything is consistent.

To show that  $s^*(y) = x$ , we recall that  $p \circ s = \text{id}$  so

$$s^*(y) = s^*(p^*(x)) = (p \circ s)^*(x) = \text{id}^*(x) = x.$$

To do the parallel thing for  $s^*(z)$ , we need to understand the composite  $q \circ s: \mathbb{CP}^1 \rightarrow \mathbb{CP}^n$ , which just sends  $L$  to  $\alpha(L)$ . This is homotopic to the standard inclusion  $\mathbb{CP}^1 \rightarrow \mathbb{CP}^n$ , as we see from the discussion in Construction 6.3 of the file `examples.pdf`, or the explicit homotopy

$$h(t, [z_0 : z_1]) = [\cos(\pi t/2)z_0 - \sin(\pi t/2)z_1 : \sin(\pi t/2)z_0 + \cos(\pi t/2)z_1 : 0 : \dots : 0].$$

It follows that  $(q \circ s)^*(x) = x$ , or equivalently  $s^*(z) = x$  as claimed.

(e) Suppose we have  $u \in H^2(M)$ , so  $u = ay + jz$  for some  $a, j \in \mathbb{Z}$ . As  $y^2 = 0$  we then have  $u^2 = 2ajyz + j^2z^2$ . As  $\{yz, z^2\}$  is a basis for  $H^4(M)$  we therefore have  $u^2 = 0$  iff  $j^2 = 2aj = 0$ , but  $a$  and  $j$  are just integers, so this happens iff  $j = 0$ . Thus, we have  $u^2 = 0 \neq u$  iff  $u = ay$  for some  $a \in \mathbb{Z} \setminus \{0\}$ .

Now suppose we have another element  $v = by + kz \in H^2(M)$  with  $v^n = 0 \neq v^2$ . As  $v^2 \neq 0$  we must have  $k \neq 0$ . As  $y^2 = 0$ , the binomial expansion just gives  $v^n = k^n z^n + nk^{n-1}bz^{n-1}y$ . However, we have  $z^n = z^{n-1}y$  in  $H^*(M)$ , so this becomes  $v^n = k^{n-1}(k + nb)z^{n-1}y$ . As  $v^n = 0$  and  $\{z^{n-1}y\}$  is a basis for  $H^{2n}(M)$  and  $k \neq 0$ , we see that  $k + nb = 0$ . Thus, we have  $v = b(y - nz)$ .

(f) Suppose we had a homotopy equivalence  $f: M \rightarrow \mathbb{CP}^1 \times \mathbb{CP}^{n-1}$ . Let  $u$  and  $v$  be the obvious generators of  $H^*(\mathbb{CP}^1 \times \mathbb{CP}^n)$ , so that

$$H^*(\mathbb{CP}^1 \times \mathbb{CP}^n) = \mathbb{Z}[u, v]/(u^2, v^n).$$

The map  $f$  would give an isomorphism

$$\mathbb{Z}[u, v]/(u^2, v^n) \rightarrow H^*(M),$$

which would give elements  $f^*(u), f^*(v) \in H^2(M)$  with  $f^*(u)^2 = 0 \neq f^*(u)$  and  $f^*(v)^n = 0 \neq f^*(v)^2$ . By part (d), we would then have  $f^*(u) = ay$  and  $f^*(v) = b(y - nz)$  for some  $a, b \in \mathbb{Z}$ . As  $f^*$  is supposed to be an isomorphism, these elements would need to generate the group  $H^2(M) = \mathbb{Z}\{y, z\}$ . However, by working mod  $n$  it is clear that  $z$  cannot be expressed as a  $\mathbb{Z}$ -linear combination of  $ay$  and  $b(y - nz)$ , so we have a contradiction.

**Problem 9.4.** Let  $M$  be the Milnor hypersurface in  $\mathbb{CP}^2 \times \mathbb{CP}^3$ , and let  $y$  and  $z$  be the standard generators of  $H^*M$ . Give a basis for  $H^*M$ , and express  $(y + z)^4$  in terms of that basis.

**Solution:** Results from lectures tell us that

$$H^*M = \mathbb{Z}[y, z]/(y^3, z^3 - yz^2 + y^2z - y^3) = \mathbb{Z}\{1, y, y^2, z, yz, y^2z, z^2, yz^2, y^2z^2\}.$$

Now consider  $(y + z)^4 \in H^8M$ . The only basis element in degree 8 is  $y^2z^2$ , so we must have  $(y + z)^4 = my^2z^2$  for some  $m \in \mathbb{Z}$ . Our problem is to find  $m$ .

The relations can be written as  $y^3 = 0$  and  $z^3 = yz^2 - y^2z + y^3 = yz^2 - y^2z$ . It follows that

$$\begin{aligned} (y + z)^3 &= y^3 + 3y^2z + 3yz^2 + z^3 \\ &= 0 + 3y^2z + 3yz^2 + yz^2 - y^2z \\ &= 2y^2z + 4yz^2 \\ (y + z)^4 &= (y + z)(y + z)^3 = (y + z)(2y^2z + 4yz^2) \\ &= 2y^3z + 6y^2z^2 + 4yz^3. \end{aligned}$$

Here we have  $y^3 = 0$  and  $z^3 = yz^2 - y^2z$  so  $4yz^3 = 4y^2z^2 - 4y^3z = 4y^2z^2$ . Putting this together, we get  $(y + z)^4 = 10y^2z^2$ .

**Problem 9.5.** Let  $M$  be the Milnor hypersurface  $H_{24}$ .

(a) For each  $i$ , find a basis  $B_i$  for the free abelian group  $H^iM$ . You should find that  $B_{10}$  consists of a single element which we will call  $u$ .

- (b) Suppose that  $B_4 = \{a_1, \dots, a_r\}$  and  $B_6 = \{b_1, \dots, b_r\}$ . Each product  $a_i b_j$  then lies in  $H^{10}M$  and so is equal to  $m_{ij}u$  for some integer  $m_{ij}$ . Find these integers.
- (c) Hence find an alternative basis  $b'_1, \dots, b'_r$  for  $H^6M$  such that

$$a_i b'_j = \begin{cases} u & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

**Solution:**

- (a) As in the notes (Proposition 16.8 in **examples.pdf**) we have

$$H^*M = \mathbb{Z}[y, z]/(y^3, z^4 - yz^3 + y^2z^2 - y^3z + y^4) = \mathbb{Z}\{y^i z^j \mid 0 \leq i \leq 2, 0 \leq j \leq 3\}.$$

In other words, the set

$$B = \{1, y, y^2, z, yz, y^2z, z^2, yz^2, y^2z^2, z^3, yz^3, y^2z^3\}$$

is a basis for the whole cohomology ring. Thus, we have bases  $B_i$  for the individual groups  $H^iM$  as follows:

$$B_0 = \{1\}$$

$$B_2 = \{y, z\}$$

$$B_4 = \{y^2, yz, z^2\}$$

$$B_6 = \{y^2z, yz^2, z^3\}$$

$$B_8 = \{y^2z^2, yz^3\}$$

$$B_{10} = \{y^2z^3\}$$

$$B_i = \emptyset \text{ for } i \notin \{0, 2, 4, 6, 8, 10\}.$$

In particular, the unique basis element for  $H^{10}M$  is  $u = y^2z^3$ .

- (b) In  $H^*M$  we have the following relations

$$y^3 = 0$$

$$z^4 = yz^3 - y^2z^2 + y^3z - y^4 = yz^3 - y^2z^2$$

$$yz^4 = y(yz^3 - y^2z^2) = y^2z^3 - y^3z^2 = y^2z^3 = u$$

$$z^5 = z(yz^3 - y^2z^2) = yz^4 - y^2z^3 = 0.$$

Now put

$$a_0 = y^2 \quad a_1 = yz \quad a_2 = z^2 \quad b_0 = y^2z \quad b_1 = yz^2 \quad b_2 = z^3$$

so  $B_4 = \{a_0, a_1, a_2\}$  and  $B_6 = \{b_0, b_1, b_2\}$ . We find that

$$\begin{bmatrix} a_0.b_0 & a_0.b_1 & a_0.b_2 \\ a_1.b_0 & a_1.b_1 & a_1.b_2 \\ a_2.b_0 & a_2.b_1 & a_2.b_2 \end{bmatrix} = \begin{bmatrix} y^4z & y^3z^2 & y^2z^3 \\ y^3z^2 & y^2z^3 & yz^4 \\ y^2z^3 & yz^4 & z^5 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} u.$$

- (c) Now put

$$b'_0 = b_0 - b_1 + b_2 = z^3 - yz^2 + y^2z$$

$$b'_1 = b_1 - b_0 = yz^2 - y^2z$$

$$b'_2 = b_0 = y^2z.$$

From the relations in (b) we get  $a_i b'_j = \delta_{ij}u$  as required. This can be found by inspection, or by the following logic: the relations in (b) can be written in matrix form as  $a^T b = Pu$ , where  $a = [a_0 \ a_1 \ a_2]$  and  $b = [b_0 \ b_1 \ b_2]$  and  $P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$  (so  $P^{-1} = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ ). We can therefore take  $b' = bP^{-1}$  to get  $ab' = Iu$ .

**Problem 9.6.** Let  $H_{2,2} = \{([z], [w]) \mid \sum_i z_i w_i = 0\}$  be the standard Milnor hypersurface in  $\mathbb{CP}^2 \times \mathbb{CP}^2$ , and let  $F_3 = \text{Flag}_3(\mathbb{C}^3)$  be the space of flags  $0 = W_0 < W_1 < W_2 < W_3 = \mathbb{C}^3$  in  $\mathbb{C}^3$ . By quoting results from the lectures, write down the cohomology rings of these spaces, and prove by pure algebra that they are isomorphic. Find a homeomorphism  $H_{2,2} \rightarrow F_3$ .

**Solution:** In lectures it was proved that  $H^*H_{2,2} = \mathbb{Z}[y, z]/(y^3, y^2 - yz + z^2)$ . We also described  $H^*F_3$  in terms of Euler classes  $x_1, x_2$  and  $x_3$ ; here we will call these  $u, v$  and  $w$  for ease of writing. The elementary symmetric functions are then

$$\sigma_1 = u + v + w \quad \sigma_2 = uv + uv + vw \quad \sigma_3 = uvw,$$

and we have  $H^*F_3 = \mathbb{Z}[u, v, w]/(\sigma_1, \sigma_2, \sigma_3)$ .

Thus, in  $H^*F_3$  we have  $w = -u - v$  and thus

$$uv + vw + wu = w(u + v) + uv = uv - (u + v)^2 = -u^2 - uv - v^2.$$

This means that in  $H^*F_3$  we have  $v^2 + uv + u^2 = 0$ , so  $v^2 = -uv - u^2$ . We also have

$$uvw = uv(-u - v) = -u^2v - uv^2 = -u^2v - u(-uv - u^2) = u^3.$$

Thus  $u^3 = 0$  in  $H^*F_3$ . All the above steps are reversible, so we actually have  $H^*F_3 = \mathbb{Z}[u, v]/(u^3, v^2 + uv + u^2)$ . There is now an evident isomorphism  $H^*F_3 \rightarrow H^*H_{2,2}$  sending  $u$  to  $y$  and  $v$  to  $-z$ .

Define  $\kappa: \mathbb{C}^3 \rightarrow \mathbb{C}^3$  by  $\kappa(z_0, z_1, z_2) = (\bar{z}_0, \bar{z}_1, \bar{z}_2)$ . This induces a homeomorphism  $\kappa: \mathbb{C}P^2 \rightarrow \mathbb{C}P^2$  with  $\kappa^2 = 1$ , and it is clear that  $(L, M) \in H_{2,2}$  iff  $L$  is orthogonal to  $\kappa(M)$ . We can thus define a homeomorphism  $f: H_{2,2} \rightarrow F_3$  by

$$f(L, M) = (0 < L < L \oplus \kappa(M) < \mathbb{C}^3) \\ f^{-1}(0 < W_1 < W_2 < \mathbb{C}^3) = (V_1, \kappa(W_2 \ominus W_1)).$$

**Problem 9.7.** Recall that a *Möbius transformation* is a map  $f$  from the Riemann sphere  $\mathbb{C} \cup \{\infty\} \simeq \mathbb{C}P^1$  to itself that can be written in the form  $f(z) = (az + b)/(cz + d)$  for some  $a, b, c, d \in \mathbb{C}$  with  $ad - bc \neq 0$ . Let  $M$  be the group of Möbius transformations.

Construct homeomorphisms

$$SL(2, \mathbb{C})/\{\pm 1\} \simeq M \simeq F_3(\mathbb{C} \cup \{\infty\}).$$

By considering  $SU(2)$ , construct an interesting map  $\mathbb{R}P^3 \rightarrow F_3(S^2)$ . By considering the Gram-Schmidt process, prove that this map is a homotopy equivalence.

**Solution:** Let  $A$  be the group of holomorphic automorphisms of the Riemann sphere  $S^2 = \mathbb{C} \cup \{\infty\}$ . Given a matrix  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in GL(2, \mathbb{C})$ , we have an automorphism  $\alpha(A) \in G$  defined by  $\alpha(A)(z) = (az + b)/(cz + d)$ . The easiest way to see that there are no problems at  $z = \infty$  is to identify  $S^2 = \mathbb{C} \cup \{\infty\}$  with  $\mathbb{C}P^1$ , so  $\alpha(A)$  becomes the map  $[z : w] \mapsto [az + bw : cz + dw]$ . It is well-known that this gives a group homomorphism  $\alpha: GL(2, \mathbb{C}) \rightarrow A$ , whose image  $M$  is called the Möbius group. (It is also well-known that  $M = A$ , but we do not need this.) The multiples of the identity matrix give a copy of  $\mathbb{C}^\times$  in  $GL(2, \mathbb{C})$  and visibly  $\alpha(\mathbb{C}^\times) = 1$ , so we get an induced map  $\alpha: PGL(2, \mathbb{C}) := GL(2, \mathbb{C})/\mathbb{C}^\times \rightarrow M$ . It is easy to check that this is bijective, and we topologise  $M$  to make it a homeomorphism. If  $A \in GL(2, \mathbb{C})$  then  $A' := \det(A)^{-1/2}A \in SL(2, \mathbb{C})$  and  $\alpha(A) = \alpha(A')$ . Moreover, we have  $SL(2, \mathbb{C}) \cap \mathbb{C}^\times = \{\pm 1\}$ . It follows easily that  $SL(2, \mathbb{C})/\{\pm 1\} = PGL(2, \mathbb{C}) = M$  as claimed. Now define  $\phi: M \rightarrow F_3(\mathbb{C} \cup \{\infty\})$  by  $\phi(\gamma) = (\gamma(0), \gamma(1), \gamma(\infty))$ , and define  $\psi: F_3\mathbb{C} \rightarrow M$  by

$$\psi(u, v, w)(z) = \frac{u(v - w) + (u - v)wz}{(v - w) + (u - v)z}.$$

One checks by direct calculation that  $\phi\psi = 1$  on  $F_3\mathbb{C} \subset F_3S^2$ . This can be extended over all of  $F_3S^2$  by putting

$$\psi([a : b], [c : d], [e : f])([z : w]) = [(ad - bc)ez + (cf - de)aw : (ad - bc)fw + (cf - de)bw].$$

(One checks that the determinant of the associated matrix is  $(ad - bc)(cf - de)(be - af)$ , and this is nonzero when  $[a : b]$ ,  $[c : d]$  and  $[e : f]$  are all distinct.) We still have  $\phi\psi = 1$ . In the other direction, suppose that  $\gamma \in M$  and put  $\delta = \psi\phi(\gamma)$ . Then  $\delta(z) = \gamma(z)$  when  $z \in \{0, 1, \infty\}$  and so  $\delta^{-1}\gamma$  fixes 0, 1 and  $\infty$ . If  $\delta^{-1}\gamma$  corresponds to the matrix  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$  then we see that  $b = 0$  and  $a + b = c + d$  and  $d = 0$  so  $A$  is a multiple of the identity and so  $\delta = \gamma$ . It follows that  $\psi\phi = 1$  and thus that  $\phi$  is a homeomorphism.

We now regard  $S^3$  as  $\{(a, c) \in \mathbb{C}^2 \mid |a|^2 + |c|^2 = 1\}$  and define a map  $\chi: S^3 \times \mathbb{R}^+ \times \mathbb{R} \rightarrow SL(2, \mathbb{C})$  by

$$\chi(a, c, r, t) = \begin{pmatrix} a & -\bar{b} \\ c & \bar{a} \end{pmatrix} \begin{pmatrix} r & t \\ 0 & r^{-1} \end{pmatrix}.$$

This is actually a homeomorphism, with inverse

$$\chi^{-1} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = (a/r, c/r, r, \bar{a}b + \bar{b}d),$$

where  $r = \sqrt{|a|^2 + |c|^2}$ . I'll leave you to work out how I derived this from Gram-Schmidt theory (compare Lemma 17 in the notes.) We see from the formula that  $\chi(-a, -c, r, t) = -\chi(a, c, r, t)$  so we get an induced homeomorphism  $S^3/\pm 1 \times \mathbb{R}^+ \times \mathbb{R} \rightarrow SL(2, \mathbb{C})/\pm 1$ . On the left hand side we have  $S^3/\pm 1 = \mathbb{R}P^3$  and the other two factors are contractible. On the right hand side we know that  $SL(2, \mathbb{C})/\pm 1 \simeq F_3(S^2)$ . We thus have  $\mathbb{R}P^3 \simeq F_3S^2$  as claimed.

**Problem 9.8.** For  $0 \leq k \leq n+1$  we regard  $\mathbb{C}^k$  as a subspace of  $\mathbb{C}^{n+1}$  in the usual way. Let  $B_n$  be the space of those flags  $0 = V_0 < \dots < V_{n+1} = \mathbb{C}^{n+1}$  in  $\mathbb{C}^{n+1}$  for which  $V_k \leq \mathbb{C}^{k+1}$  for  $k = 0, \dots, n$ . Define line bundles  $L_1, \dots, L_{n+1}$  and  $M_1, \dots, M_n$  over  $B_n$  by

$$\begin{aligned} L_{k,\underline{V}} &= V_k \ominus V_{k-1} \\ M_{k,\underline{V}} &= \mathbb{C}^{k+1} \ominus V_k \end{aligned}$$

(here  $W \ominus U$  means the orthogonal complement of  $U$  in  $W$ ). Check that  $L_k \oplus M_k = \mathbb{C} \oplus M_{k-1}$  and deduce some relations among Euler classes. Show how to regard  $B_n$  as a projective bundle over  $B_{n-1}$  and deduce a description of  $H^*B_n$ .

**Solution:** We have

$$\begin{aligned} L_{k,\underline{V}} \oplus M_{k,\underline{V}} &= (V_k \ominus V_{k-1}) \oplus (\mathbb{C}^{k+1} \ominus V_k) \\ &= \mathbb{C}^{k+1} \ominus V_{k-1} \\ &= (\mathbb{C}^{k+1} \ominus \mathbb{C}^k) \oplus (\mathbb{C}^k \ominus V_{k-1}) \\ &= \mathbb{C} \oplus M_{k-1,\underline{V}}, \end{aligned}$$

so  $L_k \oplus M_k = \mathbb{C} \oplus M_{k-1}$  as claimed. Put  $x_k = e(L_k^*)$  and  $y_k = e(M_k^*)$ . The above isomorphism gives an identity  $(t - x_k)(t - y_k) = t(t - y_{k-1})$  of Chern polynomials. By expanding this out, we get  $x_k + y_k = y_{k-1}$  and  $x_k y_k = 0$ . We can use the first equation to replace  $x_k$  by  $y_{k-1} - y_k$  and then the second equation becomes  $y_k^2 = y_k y_{k-1}$ .

Next, consider the map  $q: B_n \rightarrow B_{n-1}$  sending a flag  $(0 < V_1 < \dots < V_n < \mathbb{C}^{n+1})$  to the truncated flag  $(0 < V_1 < \dots < V_{n-1} < \mathbb{C}^n)$ . (Note that this procedure does not define a map  $\text{Flag}(\mathbb{C}^{n+1}) \rightarrow \text{Flag}(\mathbb{C}^n)$ ; we are using the condition  $V_{n-1} < \mathbb{C}^n$  in the definition of  $B_n$ .) Given a flag  $\underline{V} \in B_{n-1}$ , the flags over it in  $B_n$  biject with one-dimensional subspaces of  $\mathbb{C}^{n+1}/V_{n-1} = \mathbb{C} \oplus M_{n-1,\underline{V}}$  so we get a homeomorphism  $B_n = P(\mathbb{C} \oplus M_{n-1})$ . As  $f_{\mathbb{C} \oplus M_{n-1}}(t) = t(t - y_{n-1})$  this gives

$$H^*(B_n) = H^*(B_{n-1})[y_n]/(y_n^2 - y_{n-1}y_n) = H^*(B_{n-1})\{1, y_n\}.$$

We deduce by induction that the products  $y^\alpha = \prod_{i=1}^n y_i^{\alpha_i}$  (with  $\alpha_i \in \{0, 1\}$ ) form a basis for  $H^*B_n$  over  $\mathbb{Z}$ .

**Problem 9.9.** Let  $V$  be a complex vector space of dimension  $n$  and let  $S$  be a subset of  $\{1, \dots, n-1\}$ , say  $S = \{d_1, \dots, d_{m-1}\}$  with  $d_0 := 0 < d_1 < \dots < d_{m-1} < d_m := n$ . Let  $\text{Flag}_S(V)$  be the space of sequences  $(V_{d_1} < \dots < V_{d_m} < V)$  such that  $\dim(V_{d_k}) = d_k$  for all  $k$ . Guess a description of  $\text{Hom}(H^* \text{Flag}_S(V), R)$  for any ring  $R$ , and outline a proof that your guess is correct.

(Note that when  $S = \{k\}$  we have  $\text{Flag}_S(V) = \text{Grass}_k(V)$ , and when  $S = \{1, \dots, k\}$  we have  $\text{Flag}_S(V) = \text{Flag}_k(V)$ , and both of these cases have been covered in lectures.)

**Solution:** The set  $\text{Hom}(H^* \text{Flag}_S(V), R)$  bijects naturally with the set of lists  $(g_1, \dots, g_m)$  such that  $g_i \in R[t]$  is a monic polynomial of degree  $d_i - d_{i-1}$  and  $\prod_i g_i(t) = t^n$ . To prove this, write  $X_S(R)$  for the set described, and put  $Y_S(R) = \text{Hom}(H^* \text{Flag}_S(V), R)$ . Let  $W_i$  be the bundle over  $\text{Flag}_S(V)$  with fibres  $W_{i,\underline{V}} = V_{d_i}/V_{d_{i-1}}$  and put  $f_i(t) = f_{W_i}(t) \in H^* \text{Flag}_S(V)[t]$ . Clearly  $\underline{f} = (f_1, \dots, f_m)$  is an element of  $X(H^* \text{Flag}_S(V))$ . More generally, if  $\alpha: H^* \text{Flag}_S(V) \rightarrow R$  is a ring homomorphism we can apply  $\alpha$  to the coefficients of the polynomials  $f_i$  to get an element  $\alpha_*(\underline{f}) \in X_S(R)$ . We can now define  $\phi_{S,R}: Y_S(R) \rightarrow X_S(R)$  by  $\phi_{S,R}(\alpha) = \alpha_*(\underline{f})$ .

If  $S = \{1, \dots, n-1\}$  then  $\text{Flag}_S(V) = \text{Flag}(V)$  and our calculation of  $H^* \text{Flag}_S(V)$  shows that  $\phi_S$  is an isomorphism. We can thus try to prove that  $\phi_S$  is an isomorphism for all  $S$  by downwards induction on  $|S|$ . If  $|S| < n-1$  then there exists  $i \in \{1, \dots, m\}$  such that  $d_{i-1} + 1 \notin S$ . Define  $S' = S \cup \{d_{i-1} + 1\}$ ; one checks that  $\text{Flag}_{S'}(V)$  is the projective bundle associated to  $W_i$ , and thus that  $H^* \text{Flag}_{S'}(V) = H^* \text{Flag}_S(V)[x]/g_i(x)$ . If we let  $A_S$  be the universal example of a ring with a factorisation of  $t^n$  of the indicated type, then it is not hard to see that  $A_{S'} = A_S[x]/g_i(x)$ . We have a commutative square as follows:

$$\begin{array}{ccc} A_S & \longrightarrow & A_{S'} \\ \downarrow & & \downarrow \\ H^* \text{Flag}_S(V) & \longrightarrow & H^* \text{Flag}_{S'}(V) \end{array}$$

By induction we may assume that the right hand vertical map is an isomorphism. As  $A_{S'}$  is free over  $A_S$  and  $H^* \text{Flag}_{S'}(V)$  is free over  $H^* \text{Flag}_S(V)$  with the same set of generators, we see that the left hand vertical map is also an isomorphism.

**Problem 9.10.** Let  $X$  be the space  $\text{Flag}(\mathbb{C}^3)$  of complete flags in  $\mathbb{C}^3$ . Find an element  $u$  that generates the highest-degree nontrivial cohomology group of  $X$ . Find two bases  $a_1, \dots, a_n$  and  $b_1, \dots, b_n$  for  $H^*(X)$  such that  $a_i b_i = u$  for all  $i$  and  $a_i b_j = 0$  for all  $i \neq j$  with  $|a_i| + |b_j| = |u|$ . Show that for every  $v \in H^2(X)$  we have  $v^3 = 0 \pmod{3}$ .

**Solution:** From results in the notes we have

$$\begin{aligned} H^*(X) &= \mathbb{Z}[x, y, z]/(x + y + z, xy + xz + yz, xyz) \\ &= \mathbb{Z}[x, y]/(y^2 + xy + x^2, x^3) = \mathbb{Z}\{1, x, x^2, y, xy, x^2y\}. \end{aligned}$$

If we multiply the relation  $x^2 + xy + y^2 = 0$  by  $x$  and subtract  $x^3 = 0$  we get  $xy(x + y) = 0$ . Now put  $u = x^2y$  and define elements  $a_i$  and  $b_i$  as follows:

|       |        |            |       |         |      |        |
|-------|--------|------------|-------|---------|------|--------|
| $i$   | 1      | 2          | 3     | 4       | 5    | 6      |
| $a_i$ | 1      | $x$        | $y$   | $x^2$   | $xy$ | $x^2y$ |
| $b_i$ | $x^2y$ | $x(x + y)$ | $x^2$ | $x + y$ | $x$  | 1      |

Using the relations listed above we get  $a_i b_i = u$  for all  $i$  and also  $a_2 b_3 = a_3 b_2 = a_4 b_5 = a_5 b_4 = 0$  as required.

Next, any element  $v \in H^2(X)$  is  $v = px + qy$  for some  $p, q \in \mathbb{Z}$ . This gives

$$v^3 = p^3 x^3 + 3p^2 q x^2 y + 3pq^2 x y^2 + q^3 y^3.$$

We have seen that  $x^3 = 0$ . The relation  $xy(x + y) = 0$  can be rewritten as  $xy^2 = -x^2y$ . We can multiply the relation  $y^2 + xy + x^2 = 0$  by  $y$  and use  $xy(x + y) = 0$  to get  $y^3 = 0$ . Using these facts we get

$$v^3 = 3pq(q - p)x^2y = 0 \pmod{3}.$$

**Problem 9.11.** Give a basis for  $H^*(\text{Grass}_2(\mathbb{C}^4))$ .

**Solution:** Put  $R = \mathbb{Z}[a_1, a_2, b_1, b_2]$  (with  $|a_i| = |b_i| = 2i$ ) and

$$f(t) = (t^2 + a_1 t + a_2)(t^2 + b_1 t + b_2) \in R[t].$$

Let  $I$  be the ideal in  $R$  generated by the coefficients of  $f(t) - t^4$ , and put  $Q = R/I$ . We have proved elsewhere that  $H^*(\text{Grass}_2(\mathbb{C}^4)) \simeq Q$ ; here we just analyse  $Q$  as a problem in pure algebra.

By expanding out the definition of  $f(t)$ , we see that

$$I = (a_2 b_2, a_1 b_2 + a_2 b_1, a_2 + a_1 b_1 + b_2, a_1 + b_1).$$

Working modulo  $I$ , we first note that  $b_1 = -a_1$ . Substituting this into the other relations gives

$$\begin{aligned} a_2 b_2 &= 0 \\ a_1 b_2 - a_1 a_2 &= 0 \\ a_2 - a_1^2 + b_2 &= 0. \end{aligned}$$

The last of these relations gives  $b_2 = a_1^2 - a_2$ . Substituting this into the other relations gives

$$\begin{aligned} \text{(A)} \quad a_1^2 a_2 &= a_2^2 \\ \text{(B)} \quad a_1^3 &= 2a_1 a_2 \end{aligned}$$

We next claim that

$$\begin{aligned} \text{(C)} \quad a_1 a_2^2 &= 0 \\ \text{(D)} \quad a_2^3 &= 0. \end{aligned}$$

Indeed, if we multiply (A) by  $a_1$  and (B) by  $a_2$  and subtract then we get (C). If we then multiply (A) by  $a_2$  and use (C) we get (D). Now define the *weight* of a monomial  $a_1^i a_2^j$  to be  $i + j$ . Our relations (A) to (D) show that every monomial of weight 3 can be rewritten as a linear combination of monomials of lower weight. The list of monomials of weight less than 3 is

$$B = \{1, a_1, a_1^2, a_2, a_1 a_2, a_2^2\},$$

and it now follows that this is a basis for  $Q$ . In more detail, we have

$$\begin{aligned} Q^0 &= \mathbb{Z} \\ Q^2 &= \mathbb{Z}\{a_1\} \\ Q^4 &= \mathbb{Z}\{a_1^2, a_2\} \\ Q^6 &= \mathbb{Z}\{a_1 a_2\} \\ Q^8 &= \mathbb{Z}\{a_2^2\}. \end{aligned}$$

**Problem 9.12.** Let  $L$  be a complex line bundle over a space  $X$ , and let  $L^j$  denote the  $j$ -fold tensor power of  $L$  (eg  $L^3 = L \otimes L \otimes L$ ). Show that  $e(L \oplus L^2 \oplus L^3) = e(L \oplus L^6 \oplus L)$ .

**Solution:** Put  $x = e(L) \in H^2(X)$ . The general rule for complex line bundles is that  $e(P \otimes Q) = e(P) + e(Q)$ , which easily implies that  $e(L^j) = jx$ . The general rule for direct sums is that  $e(U \oplus V) = e(U)e(V)$ . We therefore have

$$e(L \oplus L^2 \oplus L^3) = x.(2x).(3x) = 6x^3$$

$$e(L \oplus L^6 \oplus L) = x.(6x).x = 6x^3.$$

**Problem 9.13.** Let  $M$  be a compact connected oriented manifold of dimension  $n$ , with fundamental group  $\pi_1 M = G$ . You may assume that the groups  $H_k M$  and  $H^k M$  are finitely generated and that the following three theorems hold.

- **Poincaré duality:** There is a natural isomorphism  $H_k M \rightarrow H^{n-k} M$ .
- **Universal coefficients:** There is a natural pairing  $H_k M \otimes H^k M \rightarrow \mathbb{Z}$ , which gives rise to surjective maps  $H^k M \rightarrow \text{Hom}(H_k M, \mathbb{Z})$  and  $H_k M \rightarrow \text{Hom}(H^k M, \mathbb{Z})$ . These fit into short exact sequences as follows:

$$\text{Ext}(H_{k-1} M, \mathbb{Z}) \rightarrow H^k M \rightarrow \text{Hom}(H_k M, \mathbb{Z})$$

$$\text{Ext}(H^{k+1} M, \mathbb{Z}) \rightarrow H_k M \rightarrow \text{Hom}(H^k M, \mathbb{Z})$$

- **Hurewicz:** There is a natural isomorphism  $H_1 M = G/[G, G]$  (the maximal Abelian quotient of the fundamental group). If there is an integer  $k > 1$  such that  $\pi_1 M = \dots = \pi_{k-1} M = 0$ , then there is a natural isomorphism  $H_k M = \pi_k M$ .

- (1) Suppose that  $n = 3$  and  $G$  is finite. Describe  $H_* G$  and  $H^* G$  (including the ring structure) in terms of  $G$ .
- (2) Suppose that  $n = 4$  and  $G = 0$ . Prove that  $\pi_2 M$  is a free Abelian group.

**Solution:**

- (1) As  $M$  is connected we have  $H_0 M = \mathbb{Z}$  and  $H^0 M = \mathbb{Z}$ . We also have  $H_1 M = G/[G, G]$  by the Hurewicz theorem. The Universal Coefficient Theorem gives a short exact sequence  $\text{Ext}(H_0 M, \mathbb{Z}) \rightarrow H^1 M \rightarrow \text{Hom}(H_1 M, \mathbb{Z})$ , but  $H_0 M = \mathbb{Z}$  is free so  $\text{Ext}(H_0 M, \mathbb{Z}) = 0$  and  $H_1 M = G/[G, G]$  is finite so  $\text{Hom}(H_1 M, \mathbb{Z}) = 0$  so  $H^1 M = 0$ . Poincaré duality gives  $H^2 M = H_1 M = G/[G, G]$  and  $H_2 M = H^1 M = 0$  and  $H^3 M = H_0 M = \mathbb{Z}$  and  $H_3 M = H^0 M = \mathbb{Z}$ . Thus

$$H_* M = (\mathbb{Z}, G/[G, G], 0, \mathbb{Z})$$

$$H^* M = (\mathbb{Z}, 0, G/[G, G], \mathbb{Z}).$$

In particular, if  $a, b \in H^* M$  and  $|a|, |b| > 0$  then either  $a = 0$  or  $b = 0$  or  $|a|, |b| \geq 2$  so  $|ab| > 3$  and in all cases we have  $ab = 0$ , so the ring structure is essentially trivial.

- (2) Using connectivity and the Hurewicz theorem we see that  $H_0 M = \mathbb{Z}$  and  $H_1 M = 0$  and  $\pi_2 M = H_2 M$ . As  $H_2 M$  is a finitely generated Abelian group, we can write it in the form  $T \oplus F$  where  $F$  is free and  $T$  is a finite torsion group so  $\text{Hom}(T, \mathbb{Z}) = 0$  and  $\text{Hom}(H_2 M, \mathbb{Z}) = \text{Hom}(F, \mathbb{Z})$ , which is easily seen to be a free Abelian group. As  $H_1 M = 0$ , the Universal Coefficient Theorem tells us that  $H^2 M = \text{Hom}(H_2 M, \mathbb{Z})$ , so  $H^2 M$  is free. The Poincaré duality theorem tells us that  $H_2 M = H^2 M$ , so  $\pi_2 M = H_2 M$  is free as required.