

## PROBLEMS ON ALGEBRAIC TOPOLOGY

- The 2021-22 exam consisted of problems 1.2, 7.1, 4.1, 6.2 and 7.16.
- The 2022-23 exam consisted of problems 3.1, 2.1, 9.5 and 7.14.
- The 2023-24 exam consisted of problems 3.2, 2.2, 9.10 and 7.15.
- The 2024-25 exam consisted of problems 9.3 and 1.14.

### 1. HOMEOMORPHISMS

**Problem 1.1.** Let  $x$  be a point in  $\mathbb{R}^n$ , and suppose  $\epsilon > 0$ . Put  $U = \{y \in \mathbb{R}^n \mid \|x - y\| < \epsilon\}$ . In lectures we claimed that there is a homeomorphism  $f: U \rightarrow \mathbb{R}^n$  given by

$$f(y) = \frac{y - x}{1 - \|y - x\|^2/\epsilon^2} \qquad f^{-1}(z) = x + \frac{\sqrt{\epsilon^2 + 4\|z\|^2} - \epsilon}{2\|z\|^2} \epsilon z.$$

Check carefully that these formulae give well-defined and continuous maps with the appropriate domains and ranges, and that they are inverse to each other.

**Problem 1.2.** Consider a real vector space  $V$  with inner product, and a point  $x \in S(V)$ . Put  $A = \{y \in V \mid \langle y, x \rangle = 0\}$  and  $B = \{z \in S(V) \mid \langle z, x \rangle > 0\}$ . In the notes it is stated that there is a homeomorphism  $f: A \rightarrow B$  given by  $f(y) = (x + y)/\sqrt{1 + \|y\|^2}$ . Prove carefully that this is indeed a homeomorphism, and give a formula for the inverse. [Hint: you can start by simplifying  $\langle x, f(y) \rangle$ .]

**Problem 1.3.** Recall that  $\mathfrak{u}(n) = \{\beta \in M_n(\mathbb{C}) \mid \beta + \beta^\dagger = 0\}$ . Find a basis for  $\mathfrak{u}(2)$  over  $\mathbb{R}$ , and prove that  $\mathfrak{u}(2)$  is not a complex vector subspace of  $M_2(\mathbb{C})$ .

**Problem 1.4.** Recall our definition of the lens space: we have a complex vector space  $V$  of dimension  $n$  with inner product, and an integer  $d > 1$ . We put  $C_d = \{\omega \in \mathbb{C} \mid \omega^d = 1\}$ , and we let this act on  $S(V)$  by multiplication. The lens space is then  $M = S(V)/C_d$ . What can you say in the special case where  $n = 1$ , or the special case where  $d = 2$ ?

**Problem 1.5.** Let  $V$  be a finite-dimensional vector space with inner product. In Section 4 of the notes we defined spaces  $S(V_+)$ ,  $S'(V_+)$ ,  $S_+(V_+)/S(V)$ ,  $S^V$ , and  $B(V)/S(V)$ , and gave a table of formulae giving homeomorphisms between all these spaces. Verify a few of these formulae.

**Problem 1.6.** By quoting a suitable general theorem, prove that  $\Delta_1 \times \Delta_2 \times \Delta_3$  is homeomorphic to  $\Delta_6$ .

**Problem 1.7.** Consider the square  $X = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x, y \leq 1\}$  and the edge  $Y = \{(x, 0) \mid 0 \leq x \leq 1\}$ . Prove that  $X/Y$  is homeomorphic to  $B^2$ .

**Problem 1.8.** Let  $X$  be a space, and  $Y$  a closed subspace, and let  $Z$  be any other space. Construct a continuous bijection  $f: (X/Y) \wedge Z_+ \rightarrow (X \times Z)/(Y \times Z)$ .

(In the cases of interest  $f^{-1}$  will be continuous so that  $f$  is a homeomorphism, but there are technical subtleties around this point.)

**Problem 1.9.** If  $X$  and  $Y$  are finite based sets, with  $|X| = n$  and  $|Y| = m$ , what are  $|X \vee Y|$  and  $|X \wedge Y|$ ?

**Problem 1.10.** Recall that the Cayley map  $f: \mathfrak{u}(2) \rightarrow U(2)$  is given by  $f(\beta) = (1 + \beta/2)(1 - \beta/2)^{-1}$ . Put  $X = U(2) \setminus f(\mathfrak{u}(2))$ . Prove that  $X = \{\alpha \in U(2) \mid -1 \text{ is an eigenvalue of } \alpha\}$ , and thus give a homeomorphism

$$g: (\mathbb{C}P^1 \times S^1)/(\mathbb{C}P^1 \times \{-1\}) \rightarrow X.$$

**Problem 1.11.** Let  $V$  be a finite-dimensional complex vector space equipped with a hermitian inner product. Recall that

$$U(V) = \{\alpha \in \text{End}(V) \mid \alpha^\dagger \alpha = 1\}$$

$$\mathfrak{u}(V) = \{\alpha \in \text{End}(V) \mid \alpha^\dagger + \alpha = 0\}.$$

For any  $\alpha \in \text{End}(V)$ , we put

$$\|\alpha\| = \sup\{\|\alpha(v)\| \mid v \in S(V)\}$$

$$\exp(\alpha) = \sum_{k=0}^{\infty} \frac{\alpha^k}{k!} \in \text{End}(V)$$

We also put

$$W = \{\beta \in \mathfrak{u}(V) \mid \|\beta\| < \pi\}.$$

Prove that  $\exp$  maps  $\mathfrak{u}(V)$  to  $U(V)$ . Prove also that if  $\beta \in W$  then  $\exp(\beta)$  has precisely the same eigenspaces as  $\beta$ , and deduce that  $\exp: W \rightarrow U(V)$  is injective.

When proving that  $U(V)$  is a manifold, the notes use the Cayley transform map

$$f_\alpha(\beta) = (1 + \beta/2)(1 - \beta/2)^{-1}\alpha.$$

We could instead have used the function  $\beta \mapsto \exp(\beta)\alpha$ ; this generalises more easily to other Lie groups, but is more fiddly.

**Problem 1.12.** Consider the group  $C = \{1, g\}$  with  $g^2 = 1$ . This can be made to act on  $S^2$  in three different ways, listed below. In each case identify the orbit space  $S^2/C$  and decide whether it is a manifold.

- (a)  $g \cdot [x, y, z] = [-x, y, z]$
- (b)  $g \cdot [x, y, z] = [-x, -y, z]$
- (c)  $g \cdot [x, y, z] = [-x, -y, -z]$ .

(Here (b) is significantly harder than (a) or (c). You might want to start by thinking about the analogue of (b) for  $S^1$ , and by considering  $S^1$  as a subspace of  $\mathbb{C}$ .)

**Problem 1.13.** Which of the following are manifolds?

- (a)  $M_0 = \{(x, y) \in \mathbb{R}^2 \mid xy = 0\}$
- (b)  $M_1 = \{(x, y) \in \mathbb{R}^2 \mid xy < 1\}$
- (c)  $M_2 = \{(x, y) \in \mathbb{R}^2 \mid xy \leq 1\}$
- (d)  $M_3 = \{(x, y) \in \mathbb{R}^2 \mid xy = 1\}$ .

**Problem 1.14.** Let  $U$  and  $V$  be finite-dimensional vector spaces over  $\mathbb{R}$  equipped with inner products. Suppose that  $\dim(U) = n \geq 2$  and  $\dim(V) = m \geq 1$ . We can think of  $S(U)$  as a subspace of  $S(U \oplus V)$ , and then form the complement  $S(U \oplus V) \setminus S(U)$  or the collapsed space  $S(U \oplus V)/S(U)$ .

- (a) Prove that the inclusion  $i: S(U) \rightarrow S(U \oplus V)$  is homotopic to a constant map. What does this imply about the resulting map  $i^*$  of cohomology rings?
- (b) Prove that  $S(U \oplus V) \setminus S(U)$  is homotopy equivalent to  $S(V)$ .
- (c) Find the cohomology of  $S(U \oplus V)/S(U)$  (paying attention to the special features that occur when  $m = 1$ ).
- (d) In the case where  $V = \mathbb{R}$ , describe the space  $S(U \oplus \mathbb{R})/S(U)$  more explicitly. You may wish to start by drawing pictures for the case where  $U = \mathbb{R}^2$  and  $V = \mathbb{R}$ .

## 2. HOMOTOPY

**Problem 2.1.** Recall that  $X$  is a *homotopy retract* of  $Y$  if there exist continuous maps  $X \xrightarrow{i} Y \xrightarrow{r} X$  such that the composite  $ri: X \rightarrow X$  is homotopic to the identity. By considering the effect of such maps on cohomology rings, show that the sphere  $S^4$  is not a homotopy retract of the 4-torus  $T = S^1 \times S^1 \times S^1 \times S^1$ .

**Problem 2.2.** Recall that  $X$  is a *homotopy retract* of  $Y$  if there exist continuous maps  $X \xrightarrow{i} Y \xrightarrow{r} X$  such that the composite  $ri: X \rightarrow X$  is homotopic to the identity.

Let  $U(n)$  be the group of  $n \times n$  unitary matrices, and let  $T(m)$  be the  $m$ -torus (i.e. the product of  $m$  copies of  $S^1$ ). By considering the relevant cohomology rings, prove that if  $n, m > 1$  then  $U(n)$  is not a homotopy retract of  $T(m)$  and also  $T(m)$  is not a homotopy retract of  $U(n)$ .

**Problem 2.3.** Prove that any homotopy equivalence  $f: X \rightarrow Y$  gives a bijection  $\pi_0(X) \simeq \pi_0(Y)$  (where  $\pi_0(X)$  is the set of path-components of  $X$ ). In particular, check that if  $X$  is path-connected and  $Y$  is homotopy equivalent to  $X$  then  $Y$  is path-connected.

**Problem 2.4.** Let  $X$  be the set of continuous maps  $f: S^1 \rightarrow S^1$ . Define a metric and thus a topology on  $X$ . Using winding numbers and the theory of coverings, give a homeomorphism  $X \simeq \mathbb{Z} \times S^1 \times Y$ , where  $Y = \{g: S^1 \rightarrow \mathbb{R} \mid g(1) = 0\}$ . Describe  $H^*X$ .

**Problem 2.5.** Recall that

$$\begin{aligned} \mathbb{R}^\infty &= \{x \in \prod_{k=0}^{\infty} \mathbb{R} \mid x_k = 0 \text{ for } k \gg 0\} \\ &= \bigoplus_k \mathbb{R} = \bigcup_k \mathbb{R}^k, \end{aligned}$$

with the colimit topology.

$$S^\infty = \{x \in \mathbb{R}^\infty \mid \sum_k x_k^2 = 1\} = \bigcup_k S^k.$$

Verify that the topology on  $S^\infty$  as a subspace of  $\mathbb{R}^\infty$  is the same as its topology as the colimit of the spaces  $S^k$ . Prove that  $S^\infty$  is contractible. You may like to consider the maps  $\mu(a, b, c): \mathbb{R}[t] \rightarrow \mathbb{R}[t]$  given by  $f \mapsto a + (b + ct)f$ .

### 3. COCHAIN COMPLEXES

**Problem 3.1.** Define a cochain complex  $U^*$  as follows. For all  $n \in \mathbb{Z}$ , the group  $U^n$  is generated by an element  $u_n$  with  $100u_n = 0$ , so  $U^n \simeq \mathbb{Z}/100$ . The differential  $d: U^n \rightarrow U^{n+1}$  is given by  $d(u_n) = 10u_{n+1}$ . Prove that  $H^*U = 0$  but that the identity map  $1: U^* \rightarrow U^*$  is not cochain-homotopic to zero.

**Problem 3.2.** Let  $U^* \xrightarrow{i} V^* \xrightarrow{p} W^*$  be a short exact sequence of cochain complexes and cochain maps. Suppose that all the groups  $H^k(U)$ ,  $H^k(V)$  and  $H^k(W)$  are finite, and that  $|H^j(U)|$  and  $|H^k(V)|$  are coprime for all  $j$  and  $k$ . What can you deduce about  $H^*(W)$ ? (If you make any general claims about the theory of finite abelian groups, you should give at least a brief justification.)

### 4. MAYER-VIETORIS

**Problem 4.1.** Say that a subset  $V \subseteq \mathbb{R}^n$  is *subconvex* if it is either convex or empty (so that the intersection of any two subconvex sets is subconvex). Let  $U$  be an open subset of  $\mathbb{R}^n$ , and suppose that  $U = U_1 \cup \cdots \cup U_m$  where each set  $U_i$  is open and subconvex. Prove that  $H^j(U) = 0$  for  $j \geq m$ . [Hint: you will need to use induction, but you should formulate the induction statement carefully.]

**Problem 4.2.** Put  $A = \{0, 1, \dots, n-1\} \subseteq \mathbb{R}$  and  $U = \mathbb{R}^2 \setminus (A \times \{0\})$ . Calculate  $H^*(U)$ . (Hint: consider the sets  $U_\pm = \mathbb{R}^2 \setminus (A \times [0, \pm\infty))$  and use the Mayer-Vietoris sequence.)

## 5. THE KÜNNETH THEOREM

**Problem 5.1.** Suppose we have a space  $X$  with  $H^0 X \simeq \mathbb{Z}$  and  $H^1(X) \simeq \mathbb{Z}/2$  and  $H^2(X) \simeq \mathbb{Z}/3$ . What can we say about  $H^2(X \times \mathbb{C}P^1)$ ?

**Problem 5.2.** Consider the spaces  $X = \mathbb{C} \setminus \{0, 1\}$  and  $Y = \mathbb{C} \setminus \{0, 1, 2\}$ . The cohomology of these was described in lectures. Describe  $H^n(X \times Y)$  for all  $n$ . Show that  $a^2 = 0$  for all  $a \in H^1(X \times Y)$ .

## 6. CONFIGURATION SPACES

**Problem 6.1.** Consider the space

$$X = F_4 \mathbb{C} = \{(z_0, z_1, z_2, z_3) \in \mathbb{C}^4 \mid z_i \neq z_j \text{ whenever } i \neq j\}.$$

The cohomology of  $X$  was described in lectures in terms of generators and relations. Use this to give a basis for  $H^*(X)$ . (You can check your answer against the following facts:  $H^*(F_n \mathbb{C})$  has total rank  $n!$ , whereas the group  $H^{n-1}(F_n \mathbb{C})$  has rank  $(n-1)!$ , and the groups  $H^m(F_n \mathbb{C})$  are zero for  $m \geq n$ .)

**Problem 6.2.** In the notes we defined spaces  $F_n \mathbb{C}$  and elements  $a_{ij} \in H^1(F_n \mathbb{C})$ . Prove that  $a_{ij}a_{jk}a_{ki} = 0$  in  $H^3(F_n \mathbb{C})$ . Then prove that  $a_{ij}a_{jk}a_{km}a_{mi} = 0$  in  $H^4(F_n \mathbb{C})$ . Then give an inductive proof of the obvious generalisation.

**Problem 6.3.** Recall that  $B_n \mathbb{C}$  is the set of subsets  $S \subset \mathbb{C}$  such that  $|S| = n$  (topologised as a quotient of  $F_n \mathbb{C}$ ). Prove that  $B_2 \mathbb{C}$  is homotopy equivalent to  $S^1$ .

**Problem 6.4.** Construct homeomorphisms

$$\begin{aligned} F_2 \mathbb{C} &\simeq \mathbb{C} \times \mathbb{C}^\times \\ F_3 \mathbb{C} &\simeq \mathbb{C} \times \mathbb{C}^\times \times (\mathbb{C} \setminus \{0, 1\}) \\ B_2 \mathbb{C} &\simeq \mathbb{C} \times \mathbb{C}^\times \end{aligned}$$

Describe the cohomology of all these spaces.

**Problem 6.5.** Let  $F_2 \mathbb{R}^n$  denote the space of pairs  $(a, b)$  with  $a, b \in \mathbb{R}^n$  and  $a \neq b$ . Let  $B_2 \mathbb{R}^n$  be the quotient of  $F_2 \mathbb{R}^n$  by the evident action of  $C_2$ , so  $(a, b) \sim (c, d)$  iff  $((a, b) = (c, d) \text{ or } (a, b) = (d, c))$ . Let  $\mathbb{R}P^{n-1}$  denote the space of one-dimensional subspaces  $L \leq \mathbb{R}^n$ . Show that  $B_2 \mathbb{R}^n$  is homotopy equivalent to  $\mathbb{R}P^{n-1}$ .

## 7. MATRIX GROUPS

**Problem 7.1.** If  $V$  and  $W$  are finite-dimensional complex vector spaces, we let  $\mathcal{J}(V, W)$  be the space of injective linear maps  $V \rightarrow W$ . Describe the spaces  $\mathcal{J}(\mathbb{C}, \mathbb{C})$ ,  $\mathcal{J}(\mathbb{C}, \mathbb{C}^2)$ ,  $\mathcal{J}(\mathbb{C}^2, \mathbb{C})$  and  $\mathcal{J}(\mathbb{C}^2, \mathbb{C}^2)$  up to homotopy, and describe their cohomology, including the ring structure.

**Problem 7.2.** Give a path joining  $I$  to  $-I$  in  $U(2)$ .

**Problem 7.3.** Put  $SU(n) = \{A \in U(n) \mid \det(A) = 1\}$ . Define  $\alpha: SU(3) \rightarrow S^5 \times S^5$  by  $\alpha(A) = (Ae_0, Ae_1)$  (where  $\{e_0, e_1, e_2\}$  is the standard basis of  $\mathbb{C}^3$ ). Prove that  $\alpha$  is injective but not surjective.

**Problem 7.4.** Prove that  $SU(2)$  is homeomorphic to  $S^3$ , and thus that  $U(2)$  is homeomorphic to  $S^1 \times S^3$ .

**Problem 7.5.** Prove that the space  $GL_2^+(\mathbb{R}) = \{A \in GL_2(\mathbb{R}) \mid \det(A) > 0\}$  is homeomorphic to  $\mathbb{R}^3 \times S^1$ .

**Problem 7.6.** Put  $J = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$ , and let  $G$  be the group of matrices  $A \in GL_4(\mathbb{R})$  such that  $A^T J A = J$ . This is called the *Lorenz group*. Prove that it has at least four path-components.

**Problem 7.7.** Recall the complex reflection map  $\rho: S^1 \times \mathbb{C}P^1 \rightarrow U(2)$ : the matrix  $\rho(z, L)$  has eigenvalue  $z$  on  $L$ , and eigenvalue 1 on  $L^\perp$ . Consider the following two matrices:

$$A = \frac{1}{2} \begin{bmatrix} i+1 & -i-1 \\ i+1 & i+1 \end{bmatrix} \quad B = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$$

One of these has the form  $\rho(z, L)$  for some  $z$  and  $L$ , and the other does not lie in the image of  $\rho$ . Work out which is which, and find  $z$  and  $L$ .

**Problem 7.8.** Give a formula for the rank of the free abelian group  $H^*U(n)$ .

**Problem 7.9.** Give a basis for  $\tilde{H}^*(U(4)/U(2))$ . (This should be interpreted as the space obtained from  $U(4)$  by collapsing  $U(2)$  to a point, not the coset space.)

**Problem 7.10.** Find an integer  $n$  and a class  $u \in H^*U(n)$  such that  $u^2$  is a nonzero element of  $H^{n^2}U(n)$ .

**Problem 7.11.** Let  $\mu: U(3) \times U(3) \rightarrow U(3)$  be given by  $\mu(A, B) = AB$ . By quoting facts about  $\mu^*$  proved in lectures, calculate  $\mu^*(a_1 a_3 a_5) \in H^9(U(3) \times U(3))$ .

**Problem 7.12.** Exhibit a homeomorphism  $U(n) = S^1 \times SU(n)$ , and check that it is not an isomorphism of groups unless  $n = 1$ . Prove that

$$H^*SU(n) = (H^*U(n))/a_1 = E[a_3, a_5, \dots, a_{2n-1}].$$

**Problem 7.13.** It is known that any manifold  $M$  can be embedded as a subspace of a finite dimensional vector space. As an example, exhibit an embedding of  $PV$  in  $\text{Hom}(V, V)$  (for any Hermitian space  $V$ ). (Look through the discussion of the topology of  $U(V)$  for hints.)

**Problem 7.14.** For a complex matrix  $A$  we write  $A^\dagger$  for the conjugate transpose, so  $A_{ij}^\dagger = \overline{A_{ji}}$ . Recall that  $SU(2)$  is the group of  $2 \times 2$  complex matrices  $A$  such that  $A^\dagger A = I$  and  $\det(A) = 1$ .

- Consider a matrix  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_2\mathbb{C}$ . In terms of the entries  $a, b, c$  and  $d$ , what are the conditions for  $A$  to be an element of  $SU(2)$ ?
- Show that if  $A \in SU(2)$  and the first column of  $A$  is  $e_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$  then  $A = I$ . Deduce that if  $A, B \in SU(2)$ , and  $A$  and  $B$  have the same first column, then  $A = B$ .
- Show that there is a homeomorphism  $f: S^3 \rightarrow SU(2)$  given by

$$f((w, x, y, z)) = \begin{bmatrix} w + ix & -y + iz \\ y + iz & w - ix \end{bmatrix}.$$

(Do not assume any properties of this formula without checking them.) Note that this means that  $S^3$  can be identified with  $SU(2)$  and so has a group structure.

- Now put

$$F_3 S^3 = \{(u, v, w) \in S^3 \times S^3 \times S^3 \mid u \neq v, u \neq w, v \neq w\}$$

Prove that this space is homeomorphic to  $S^3 \times \mathbb{R}^3 \times (\mathbb{R}^3 \setminus \{0\})$  and so is homotopy equivalent to  $S^3 \times S^2$ .

(**Hint:** use the group structure from (a), and stereographic projection.)

- Deduce a description of the cohomology of  $F_3 S^3$ .

**Problem 7.15.** Let  $X \subset U(3)$  be the space of  $3 \times 3$  unitary matrices whose characteristic polynomial is  $t^3 - t^2 - t + 1$ . Find a well-known space  $Y$  that is homeomorphic to  $X$  and thus describe  $H^*(X)$ . Your answer should specify the topologies used on  $X$  and  $Y$  and explain why relevant maps are continuous.

(Hint: unitary matrices are diagonalisable, and eigenvectors with distinct eigenvalues are orthogonal.)

**Problem 7.16.** Exhibit a homeomorphism  $f: S^1 \times SU(3) \rightarrow U(3)$ . Deduce that

$$\text{rank}(H^i(U(3))) = \text{rank}(H^i(SU(3))) + \text{rank}(H^{i-1}(SU(3))),$$

and thus calculate the ranks of all the cohomology groups of  $SU(3)$ . (You may assume that these are all free abelian groups.)

## 8. VECTOR BUNDLES

**Problem 8.1.** If  $T$  is the tautological line bundle over  $\mathbb{C}P^n$ , prove that  $S(T \otimes T) = \mathbb{R}P^{2n+1}$ .

**Problem 8.2.** Let  $q: V \setminus \{0\} \rightarrow PV$  be the usual quotient map, and let  $T$  denote the tautological bundle over  $PV$ . Prove that  $q^*T$  is isomorphic to a constant bundle.

**Problem 8.3.** Let  $L$  be the Möbius bundle over  $S^1$ , given by  $L_z = \{w \in \mathbb{C} \mid w^2 \in z \cdot [0, \infty)\}$ . Prove that the bundle  $\mathbb{C} \otimes_{\mathbb{R}} L \simeq L \oplus L$  is isomorphic to a constant bundle.

**Problem 8.4.** Let  $V$  and  $W$  be complex vector bundles over a base  $X$ , and suppose that  $\dim_{\mathbb{C}}(W) = 1$ . Prove that  $P(V \otimes W) \simeq PV$ .

**Problem 8.5.** Let  $V$  be a Hermitian space, and let  $L$  be the tautological bundle over  $PV$ . Interpret  $\text{Hom}(L, L^\perp)$  as a bundle over  $PV$ , and show that it is isomorphic to the tangent bundle.

**Problem 8.6.** Let  $V$  be a complex vector bundle over  $U(n)$  that can be written as a direct sum of line bundles. What can you say about  $f_V(t)$ ?

**Problem 8.7.** Find a partition of unity on  $\mathbb{R}$  subordinate to the covering by open sets  $U_n = (n-2, n+2)$  for  $n \in \mathbb{Z}$ .

**Problem 8.8.** Consider the lens space  $X = S^3/C_d$ ; what is  $|\text{Pic}(X)|$ ?

**Problem 8.9.** Consider the  $n$ -torus  $T^n = S^1 \times \cdots \times S^1 = (S^1)^n$ . What is the rank of the abelian group  $\text{Pic}(T^n)$ ?

**Problem 8.10.** Let  $L$  be the tautological line bundle over  $\mathbb{C}P^1$ , fix an integer  $m$ , and put  $X = P(\mathbb{C} \oplus L^m)$ . For which  $m$  does there exist  $a \in H^2(X)$  such that  $a^2$  generates  $H^4(X)$ ?

## 9. MISCELLANEOUS

**Problem 9.1.** Recall that the Möbius band can be described as

$$M = \{(z, w) \in \mathbb{C} \times \mathbb{C} \mid |z| = 1 \text{ and } w^2 = tz \text{ for some } t \geq 0\}.$$

Prove that this is homotopy equivalent to  $S^1$ .

**Problem 9.2.** Let  $G$  be a path-connected topological group, such that  $H^*(G)$  is a finitely generated free abelian group. Prove that every element of  $H^1(G)$  is primitive.

**Problem 9.3.** Consider an integer  $n \geq 1$  and the Milnor hypersurface  $M = H_{1,n} \subset \mathbb{C}P^1 \times \mathbb{C}P^n$  with its two projections

$$\mathbb{C}P^1 \xleftarrow{p} M \rightarrow \mathbb{C}P^n.$$

- (a) Describe the cohomology ring  $H^*(M)$  as explicitly as possible. Your description should include the ring structure and a basis for each of the individual groups  $H^i(M)$ .

- (b) Construct a map  $s: \mathbb{CP}^1 \rightarrow M$  with  $p \circ s = \text{id}: \mathbb{CP}^1 \rightarrow \mathbb{CP}^1$ . Explain carefully why your definition is valid.
- (c) Show that when  $n = 1$  you also have  $s \circ p = \text{id}: H_{1,1} \rightarrow H_{1,1}$ , so  $s$  is a homeomorphism.
- (d) Describe the ring map  $s^*: H^*(M) \rightarrow H^*(\mathbb{CP}^1)$ , in terms of the elements that you listed in part (a). (You will need to consider properties of the map  $q \circ s: \mathbb{CP}^1 \rightarrow \mathbb{CP}^n$ .) Explain why your description is compatible with the relations in  $H^*(M)$  and  $H^*(\mathbb{CP}^1)$ , being careful to cover the case  $n = 1$  as well as the case  $n > 1$ .
- (e) Now suppose that  $n > 1$ . Suppose we have  $u, v \in H^2(M)$  with  $u^2 = 0$  and  $v^n = 0$  but  $u \neq 0$  and  $v^2 \neq 0$ . Show that there are integers  $a, b \neq 0$  with  $u = ay$  and  $v = b(y - nz)$ .
- (f) Deduce from (e) that when  $n > 1$ ,  $M$  is not homotopy equivalent to  $\mathbb{CP}^1 \times \mathbb{CP}^{n-1}$ .

**Problem 9.4.** Let  $M$  be the Milnor hypersurface in  $\mathbb{CP}^2 \times \mathbb{CP}^3$ , and let  $y$  and  $z$  be the standard generators of  $H^*M$ . Give a basis for  $H^*M$ , and express  $(y + z)^4$  in terms of that basis.

**Problem 9.5.** Let  $M$  be the Milnor hypersurface  $H_{24}$ .

- (a) For each  $i$ , find a basis  $B_i$  for the free abelian group  $H^i M$ . You should find that  $B_{10}$  consists of a single element which we will call  $u$ .
- (b) Suppose that  $B_4 = \{a_1, \dots, a_r\}$  and  $B_6 = \{b_1, \dots, b_r\}$ . Each product  $a_i b_j$  then lies in  $H^{10} M$  and so is equal to  $m_{ij} u$  for some integer  $m_{ij}$ . Find these integers.
- (c) Hence find an alternative basis  $b'_1, \dots, b'_r$  for  $H^6 M$  such that

$$a_i b'_j = \begin{cases} u & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

**Problem 9.6.** Let  $H_{2,2} = \{([\underline{z}], [\underline{w}]) \mid \sum_i z_i w_i = 0\}$  be the standard Milnor hypersurface in  $\mathbb{CP}^2 \times \mathbb{CP}^2$ , and let  $F_3 = \text{Flag}_3(\mathbb{C}^3)$  be the space of flags  $0 = W_0 < W_1 < W_2 < W_3 = \mathbb{C}^3$  in  $\mathbb{C}^3$ . By quoting results from the lectures, write down the cohomology rings of these spaces, and prove by pure algebra that they are isomorphic. Find a homeomorphism  $H_{2,2} \rightarrow F_3$ .

**Problem 9.7.** Recall that a *Möbius transformation* is a map  $f$  from the Riemann sphere  $\mathbb{C} \cup \{\infty\} \simeq \mathbb{CP}^1$  to itself that can be written in the form  $f(z) = (az + b)/(cz + d)$  for some  $a, b, c, d \in \mathbb{C}$  with  $ad - bc \neq 0$ . Let  $M$  be the group of Möbius transformations.

Construct homeomorphisms

$$SL(2, \mathbb{C})/\{\pm 1\} \simeq M \simeq F_3(\mathbb{C} \cup \{\infty\}).$$

By considering  $SU(2)$ , construct an interesting map  $\mathbb{RP}^3 \rightarrow F_3(S^2)$ . By considering the Gram-Schmidt process, prove that this map is a homotopy equivalence.

**Problem 9.8.** For  $0 \leq k \leq n + 1$  we regard  $\mathbb{C}^k$  as a subspace of  $\mathbb{C}^{n+1}$  in the usual way. Let  $B_n$  be the space of those flags  $0 = V_0 < \dots < V_{n+1} = \mathbb{C}^{n+1}$  for which  $V_k \leq \mathbb{C}^{k+1}$  for  $k = 0, \dots, n$ . Define line bundles  $L_1, \dots, L_{n+1}$  and  $M_1, \dots, M_n$  over  $B_n$  by

$$\begin{aligned} L_{k, \underline{V}} &= V_k \ominus V_{k-1} \\ M_{k, \underline{V}} &= \mathbb{C}^{k+1} \ominus V_k \end{aligned}$$

(here  $W \ominus U$  means the orthogonal complement of  $U$  in  $W$ ). Check that  $L_k \oplus M_k = \mathbb{C} \oplus M_{k-1}$  and deduce some relations among Euler classes. Show how to regard  $B_n$  as a projective bundle over  $B_{n-1}$  and deduce a description of  $H^* B_n$ .

**Problem 9.9.** Let  $V$  be a complex vector space of dimension  $n$  and let  $S$  be a subset of  $\{1, \dots, n - 1\}$ , say  $S = \{d_1, \dots, d_{m-1}\}$  with  $d_0 := 0 < d_1 < \dots < d_{m-1} < d_m := n$ . Let  $\text{Flag}_S(V)$  be the space of sequences  $(V_{d_1} < \dots < V_{d_m} < V)$  such that  $\dim(V_{d_k}) = d_k$  for all  $k$ . Guess a description of  $\text{Hom}(H^* \text{Flag}_S(V), R)$  for any ring  $R$ , and outline a proof that your guess is correct.

(Note that when  $S = \{k\}$  we have  $\text{Flag}_S(V) = \text{Grass}_k(V)$ , and when  $S = \{1, \dots, k\}$  we have  $\text{Flag}_S(V) = \text{Flag}_k(V)$ , and both of these cases have been covered in lectures.)

**Problem 9.10.** Let  $X$  be the space  $\text{Flag}(\mathbb{C}^3)$  of complete flags in  $\mathbb{C}^3$ . Find an element  $u$  that generates the highest-degree nontrivial cohomology group of  $X$ . Find two bases  $a_1, \dots, a_n$  and  $b_1, \dots, b_n$  for  $H^*(X)$  such that  $a_i b_i = u$  for all  $i$  and  $a_i b_j = 0$  for all  $i \neq j$  with  $|a_i| + |b_j| = |u|$ . Show that for every  $v \in H^2(X)$  we have  $v^3 = 0 \pmod{3}$ .

**Problem 9.11.** Give a basis for  $H^*(\text{Grass}_2(\mathbb{C}^4))$ .

**Problem 9.12.** Let  $L$  be a complex line bundle over a space  $X$ , and let  $L^j$  denote the  $j$ -fold tensor power of  $L$  (eg  $L^3 = L \otimes L \otimes L$ ). Show that  $e(L \oplus L^2 \oplus L^3) = e(L \oplus L^6 \oplus L)$ .

**Problem 9.13.** Let  $M$  be a compact connected oriented manifold of dimension  $n$ , with fundamental group  $\pi_1 M = G$ . You may assume that the groups  $H_k M$  and  $H^k M$  are finitely generated and that the following three theorems hold.

- **Poincaré duality:** There is a natural isomorphism  $H_k M \rightarrow H^{n-k} M$ .
- **Universal coefficients:** There is a natural pairing  $H_k M \otimes H^k M \rightarrow \mathbb{Z}$ , which gives rise to surjective maps  $H^k M \rightarrow \text{Hom}(H_k M, \mathbb{Z})$  and  $H_k M \rightarrow \text{Hom}(H^k M, \mathbb{Z})$ . These fit into short exact sequences as follows:

$$\text{Ext}(H_{k-1} M, \mathbb{Z}) \rightarrow H^k M \rightarrow \text{Hom}(H_k M, \mathbb{Z})$$

$$\text{Ext}(H^{k+1} M, \mathbb{Z}) \rightarrow H_k M \rightarrow \text{Hom}(H^k M, \mathbb{Z})$$

- **Hurewicz:** There is a natural isomorphism  $H_1 M = G/[G, G]$  (the maximal Abelian quotient of the fundamental group). If there is an integer  $k > 1$  such that  $\pi_1 M = \dots = \pi_{k-1} M = 0$ , then there is a natural isomorphism  $H_k M = \pi_k M$ .

- (1) Suppose that  $n = 3$  and  $G$  is finite. Describe  $H_* G$  and  $H^* G$  (including the ring structure) in terms of  $G$ .
- (2) Suppose that  $n = 4$  and  $G = 0$ . Prove that  $\pi_2 M$  is a free Abelian group.